

In this chapter we generate a "pulse train" from an arbitrary pulse to make a periodic function. We continue with the Fourier Integral notions of Chapter 1 -- such as the spectral components $X(\omega)$ -- and we then make the connection with traditional Fourier Series and their coefficients. We show how the Fourier Integral spectrum becomes discrete for a periodic function, and we are able then to relate the Fourier Series coefficients to the spectral components $X_{\text{pulse}}(\omega)$ of the pulse we use to generate our pulsetrain.

In the background and to serve as a vehicle for doing a few calculations, we address a particular problem. We consider the symmetric zero-DC-offset square-wave pulsetrain generated from two completely different methods, one involving adding a negative DC offset to a simple positive square wave pulsetrain, and the other using biphase pulses.

As noted in Chapter 1, our main purpose here is to build tools that will be used in more complicated problems. The methods presented here form the basis for treating pulse trains made from pulses of arbitrary shape, including the "arbitrariness" of a pulse being statistically present or absent. Later on this will lead to some statistical conclusions and comments about serial scramblers.

14. The Spectrum of a Pulse Train.

Let $X_{\text{pulse}}(\omega)$ be the spectrum of an arbitrary reasonable function $x_{\text{pulse}}(t)$. This $x(t)$ does not really have to be a "pulse", but it is convenient to think of it as such. We imagine that $x_{\text{pulse}}(t)$ is a function that is somewhat localized in the region of $t=0$, and vanishes for very large positive or negative time. Thus, our $x_{\text{pulse}}(t)$ is not itself periodic, and we thus expect *it* to have a continuous spectrum.

For example, from (9.2) we already know the spectrum of a single square wave pulse of height A and width τ centered at $t=0$:

$$X_{\text{pulse}}(\omega) = (A\tau) \text{sinc}(\omega\tau/2) \quad (14.1)$$

Now consider a second pulse which is a copy of our original pulse, but which is translated T_1 units to the right in time. (See footnote about T_1)¹

According to (12.1), we know the spectrum of this second pulse:

$$X(\omega, \text{second pulse}) = X_{\text{pulse}}(\omega)e^{-i\omega T_1}$$

¹ We attach a subscript 1 here because T_1 is the period of the fundamental frequency of the periodic function we are going to build. In terms of T_1 we will define $\omega_1 = 2\pi/T_1$, the corresponding angular frequency, and also $f_1 = 1/T_1$ as the non-angular frequency. Most books just use T , ω and f for these quantities. In our context, this would cause massive confusion because we are using ω (and hence its implied T and f) as our basic continuous Fourier Integral transform variable.

Now construct the infinite periodic wave by superposing pulses at $t = 0, \pm T_1, \pm 2T_1$, etc. We get:

$$X(\omega) = X_{\text{pulse}}(\omega) \sum_{n=-\infty}^{\infty} e^{i\omega(nT_1)} \quad (14.2)$$

According to our exponential sum rule (13.2) with $k = \omega T_1$, we can write the exponential sum as a sum of delta functions:

$$X(\omega) = X_{\text{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi \delta(\omega T_1 - 2\pi m) \quad (14.3)$$

In getting to the next step, we use the property of delta functions $\delta(ax) = \delta(x)/a$, and we define $\omega_1 = 2\pi/T_1$ = the fundamental angular frequency of our periodic waveform. The result is:

$$X(\omega) = X_{\text{pulse}}(\omega) \sum_{m=-\infty}^{\infty} \delta\left(\frac{\omega}{\omega_1} - m\right) \quad (14.4)$$

Notice that the delta function argument is dimensionless. This is a nice form in theory, but in practice we want ω inside the delta function to be unencumbered with any constant. The reason for this is so that when (14.4) is inserted into a $d\omega$ integration, we don't pick up any new constants. Thus, we extract the $\omega_1 = 2\pi/T_1$ factor, keeping the 2π part right with $\delta()$:

$$X(\omega) = \sum_{m=-\infty}^{\infty} (1/T_1) X_{\text{pulse}}(\omega) 2\pi \delta(\omega - m\omega_1) \quad (14.5)$$

Thus, we have a set of evenly spaced delta function spikes which occur at these frequencies:

$$\omega_m = m\omega_1 \quad m = 0, 1, 2, 3, \dots \quad (14.6)$$

In formulas like (14.5), one should keep in mind that the variable ω appearing inside $X_{\text{pulse}}(\omega)$ can at any time be replaced with $\omega_m = m\omega_1$, due to the presence of the delta function. In general, one has,

$$f(\omega) \delta(\omega - a) = f(a) \delta(\omega - a)$$

Both sides are zero when $\omega \neq a$, and at $\omega = a$, $f(a) = f(\omega)$.

Because the quantity $(1/T_1)X_{\text{pulse}}(\omega)$ is going to occur very frequently in the following discussion, we shall now define a more compact notation for it as follows:

$$c(\omega) = (1/T_1)X_{\text{pulse}}(\omega) \quad (14.7)$$

Thus, $c(\omega)$ is nothing more than our (continuous) pulse spectrum divided by the fundamental period T_1 . Thus, we can rewrite (14.5) as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega) 2\pi \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c(\omega_m) 2\pi \delta(\omega - m\omega_1) \quad (14.8)$$

As was just noted above, we can harmlessly replace ω with $\omega_m = m\omega_1$ inside $c(\omega)$ in (14.8). This leads us to define a set of numbers as follows

$$c_m + c(\omega_m) = c(m\omega_1) \quad (14.9)$$

These numbers are just the values that the function $c(\omega)$ takes at our delta spike frequencies. We arrive then at our final form for the spectrum of a pulse train,

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m 2\pi \delta(\omega - m\omega_1) \quad (14.10)$$

Now we are ready to summarize all these results:

Fourier Integral Transform of a Pulse Train

1. Let $x_{\text{pulse}}(t)$ be any reasonable pulse. Construct a pulse train $x(t)$ with spacing T_1 :

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\text{pulse}}(t - nT_1)$$

By its construction, $x(t)$ is periodic with period T_1 , which we can write formally as:

$$x(t + nT_1) = x(t) \quad n = \text{any integer}$$

If $x(t)$ is a *known* periodic function of period T_1 , a candidate for $x_{\text{pulse}}(t)$ is $x(t)$ over any one period.

2. Define $c(\omega)$ to be the Fourier Integral transform of the pulse, scaled by $1/T_1$:

$$c(\omega) + (1/T_1)X_{\text{pulse}}(\omega) = (1/T_1) \int_{-\infty}^{\infty} dt x_{\text{pulse}}(t) e^{-i\omega t}$$

3. Then the Fourier Integral transform of the Pulse *Train* is as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega) 2\pi \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m 2\pi \delta(\omega - m\omega_1)$$

where $c_m = c(\omega_m)$, $\omega_m = m\omega_1$, $\omega_1 = 2\pi/T_1$.

4. These c_m are the same c_m which appear in the the next section. That is, they are the complex Fourier Series coefficients.

Item 3 is our main result. It says that the Fourier Transform spectrum of an infinite sequence of pulses is a sum of equally-spaced delta function spikes whose coefficients are given by the continuous spectrum of the central pulse evaluated at the spike frequencies $\omega = m\omega_1$. The pulse spectrum $c(\omega) + (1/T_1)X_{\text{pulse}}(\omega)$ is normally thought of as the "coefficient envelope", while the equally spaced delta function spikes are the "lines".

It may happen that certain c_m vanish, meaning that such lines are not present.

The item 3 sum includes the DC line $m=0$ having $\omega_0 = 0$. Unless c_0 happens to vanish, the pulse train has a DC component. As noted in Section 10 above, $X_{\text{pulse}}(0)$ is the area under $x_{\text{pulse}}(t)$. Only if this area is zero, do we get $c_0 = c(0) + (1/T_1)X_{\text{pulse}}(0) = 0$.

Whereas the Fourier Transform spectrum of a single pulse (localized, non-periodic) is continuous in ω , that of an infinite sequence of pulses is entirely discrete and has no continuous portions. This conforms with the well-known fact that the spectrum of any periodic function is discrete. In fact, we have just proven this to be so.

Note on $x_{\text{pulse}}(t)$. In our summary box above, we say that if $x(t)$ is some *known* periodic function, one can take as a *candidate* for $x_{\text{pulse}}(t)$ the function $x(t)$ restricted to any one period. In this case, the dt integration endpoints for the projection $X_{\text{pulse}}(\omega)$ only cover that selected period. If we select the period centered at $t=0$, then item #2 in the above summary box becomes perhaps more familiar:

$$c(\omega) + (1/T_1)X_{\text{pulse}}(\omega) = (1/T_1) \int_{-T_1/2}^{T_1/2} dt x(t) e^{-i\omega t}$$

What is perhaps less obvious is that there are really many different candidates for $x_{\text{pulse}}(t)$ that result in the *same* $x(t)$ pulsetrain. These other choices for $x_{\text{pulse}}(t)$ are pulses which slop over into more than one period T_1 . When you form a pulsetrain with such pulses, the pulses overlap.

To see how this might work, think of a pulse which has a nice gaussian shape and goes about half way into each neighboring T_1 interval. Draw some of these, then add them up to make the sum curve $x(t)$. In this case, for your candidate $x_{\text{pulse}}(t)$, you can use either the gaussian, which overlaps into several intervals, or you can use one interval's worth of the sum curve $x(t)$.

We have tried to keep our formulas completely general to allow for pulse trains formed from pulses which overlap into more than one period. This note applies to all the analysis and summary boxes which follow.

15. Connection with the traditional Fourier Series.

If we take our pulse train spectrum (14.10) and jam it into the Fourier integral expansion (1.2), the delta functions kill the $d\omega$ integration, and we get an expansion for $x(t)$ as follows:

$$x(t) = \sum_{m=-\infty}^{\infty} c_m e^{+im\omega_1 t} \quad (15.1)$$

If we now separate out the $m=0$ term (the DC component) and then reflect the negative m values, making use of the reflection rule (7.1), we get,

$$x(t) = c_0 + 2 \operatorname{Re} \left[\sum_{m=1}^{\infty} c_m e^{+im\omega_1 t} \right] \quad (15.2)$$

We know that the c_m are in general complex numbers, so make the following two definitions:

$$\begin{aligned} a_m + 2 \operatorname{Re} [c_m] &= (2/T_1) \operatorname{Re} [X_{\text{pulse}}(m\omega_1)] \\ -b_m + 2 \operatorname{Im} [c_m] &= (2/T_1) \operatorname{Im} [X_{\text{pulse}}(m\omega_1)] \end{aligned} \quad (15.3)$$

From the way the a and b are defined in (15.3), we conclude that

$$c_m = (1/2) [a_m - ib_m] \quad (15.4)$$

If $x(t)$ is real, which we shall assume from here on out, we know from (7.1) that $X(-\omega) = X(\omega)^*$, so $X(0)$ must be real. We also know this from the "area rule" (10.1) -- the area under a real function $x(t)$ had better be real. Thus, $b_0 = 0$ and $c_0 = a_0/2$. In this case, we may write,

$$\text{DC component of } x(t) = c_0 = (a_0/2) = (1/T_1) X_{\text{pulse}}(0) \quad (15.5)$$

If we install expression (15.4) for c_m into (15.2) we get this result:

$$x(t) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos(m\omega_1 t) + \sum_{m=1}^{\infty} b_m \sin(m\omega_1 t) \quad (15.6)$$

This is recognized as the traditional Fourier Series expansion of a periodic function. Thus, our seemingly uninteresting a and b coefficients are exactly the standard Fourier Series coefficients. Moreover, the DC component of x(t) is equal to $c_0 = (a_0/2)$.

For completeness, we might as well write down an alternate form of (15.6),

$$x(t) = a_0/2 + \sum_{m=1}^{\infty} A_m \cos(m\omega_1 t + \phi_m) \quad (15.7)$$

Using $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we find that,

$$\begin{aligned} a_m &= A_m \cos(\phi_m) & A_m &= \sqrt{a_m^2 + b_m^2} & m &= 1, 2, 3, \dots \\ -b_m &= A_m \sin(\phi_m) & \tan(\phi_m) &= -b_m/a_m. \end{aligned} \quad (15.8)$$

We can regard equation (15.1) as the Fourier Series expansion formula with complex coefficients c_m . We can easily derive an expression for the corresponding Fourier Series projection formula from the bits and pieces we already have. As shown in the box in Section 14, $c_m = (1/T_1) X_{\text{pulse}}(m\omega_1)$. Using (1.1), we conclude that

$$c_m = (1/T_1) \int_{-\infty}^{\infty} dt x_{\text{pulse}}(t) e^{-im\omega_1 t} \quad (15.9)$$

By expanding the exponential inside as $\cos - i \sin$, and also expanding $c_m = (1/2) [a_m - ib_m]$ from (15.4), we get at once the traditional formulas for the Fourier coefficients, see the box below.

So, here is a summary of the above efforts:

Fourier Series Transform	
1. Let $x_{\text{pulse}}(t)$ be any reasonable pulse. Construct a pulse train $x(t)$ with spacing T_1:	
$x(t) = \sum_{n=-\infty}^{\infty} x_{\text{pulse}}(t - nT_1)$	
By its construction, $x(t)$ is periodic with period T_1, which we can write formally as:	
$x(t + nT_1) = x(t)$	$n = \text{any integer}$

If $x(t)$ is a *known* periodic function of period T_1 , a candidate for $x_{\text{pulse}}(t)$ is $x(t)$ over any one period.

2. Define the Fourier Series coefficients by these projections = transforms:

$$a_m + (2/T_1) \int_{-\infty}^{\infty} dt x_{\text{pulse}}(t) \cos(m\omega_1 t) \quad m = 0, 1, 2, 3, \dots$$

$$b_m + (2/T_1) \int_{-\infty}^{\infty} dt x_{\text{pulse}}(t) \sin(m\omega_1 t) \quad m = 1, 2, 3, \dots$$

$$c_m + (1/T_1) \int_{-\infty}^{\infty} dt x_{\text{pulse}}(t) e^{-im\omega_1 t} \quad m = \text{any integer}$$

where $c_m = (1/2) [a_m - ib_m]$.

3. The pulse train is then given by these expansions = inversions:

$$x(t) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos(m\omega_1 t) + \sum_{m=1}^{\infty} b_m \sin(m\omega_1 t) = \sum_{m=-\infty}^{\infty} c_m e^{+im\omega_1 t}$$

Thus, we have in effect "derived" the Fourier Series from our Fourier Integral theory. Notice that it is not necessary that $x_{\text{pulse}}(t)$ be totally contained within a width T_1 . We have infinite endpoints on the dt integrations above, and $x_{\text{pulse}}(t)$ is allowed to be any "reasonable" function, ie, the integrals must converge. In particular, we are allowing pulse function shapes which overlap when you make them into a pulse train.

16. Fourier Series for positive square wave pulse train.

From equation (9.2) we know the Fourier Integral spectrum for our positive square pulse of width τ and height A , namely,

$$X_{\text{pulse}}(\omega) = (A\tau) \text{sinc}(\omega\tau/2) \quad (16.1)$$

From (14.7) we find the complex Fourier Series coefficients to be,

$$c_m = (1/T_1) X_{\text{pulse}}(m\omega_1) = (A\tau/T_1) \text{sinc}(m\pi\tau/T_1)$$

Thus, from (15.3), we know the a and b coefficients as well:

$$\begin{aligned} a_m &= (2A\tau/T_1) \text{sinc}(m\pi\tau/T_1) & m = 0, 1, 2, 3, \dots \\ b_m &= 0. & m = 0, 1, 2, 3, \dots \end{aligned} \quad (16.2)$$

We have here the Fourier Series coefficients for an infinite pulse train of positive pulses of amplitude A , width τ , and period T_1 , such that the time $t=0$ occurs in the middle of a positive pulse.

The reader is invited to compute the above Fourier Series coefficients in the standard manner, using the conventional formulas in the above summary box. This is done also on page 32 of Bennet and Davey, *Data Transmission*, McGraw-Hill, (1965) [BTS Library]. Their result agrees with the above.

17. More about positive square-wave pulse trains.

We can now summarize what we know about the positive square wave pulse train:

$$x_{\text{pulse}}(t) = A [\theta(t + \tau/2) - \theta(t - \tau/2)]. \quad (17.1)$$

$$c(\omega) = (1/T_1) X_{\text{pulse}}(\omega) = (A\tau/T_1) \text{sinc}(\omega\tau/2) \quad (17.2)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\text{pulse}}(t - nT_1) \quad (17.3)$$

$$X(\omega) = \text{Error!} \quad (17.4)$$

$$c_m = (1/T_1) X_{\text{pulse}}(m\omega_1) = (A\tau/T_1) \text{sinc}(m\pi\tau/T_1) \quad (17.5)$$

$$\text{DC component} = c_0 = (A\tau/T_1) \quad (17.6)$$

where we use $\omega_1 = 2\pi/T_1$.

For general τ , all spectral lines are present. Apart from an overall constant, the envelope function $c(\omega)$ is $\text{sinc}(x)$, where $x = \omega\tau/2$. Here is a linear graph of $|\text{sinc}(x)| = |\sin(x)/x|$ along with a graph of $1/x$:

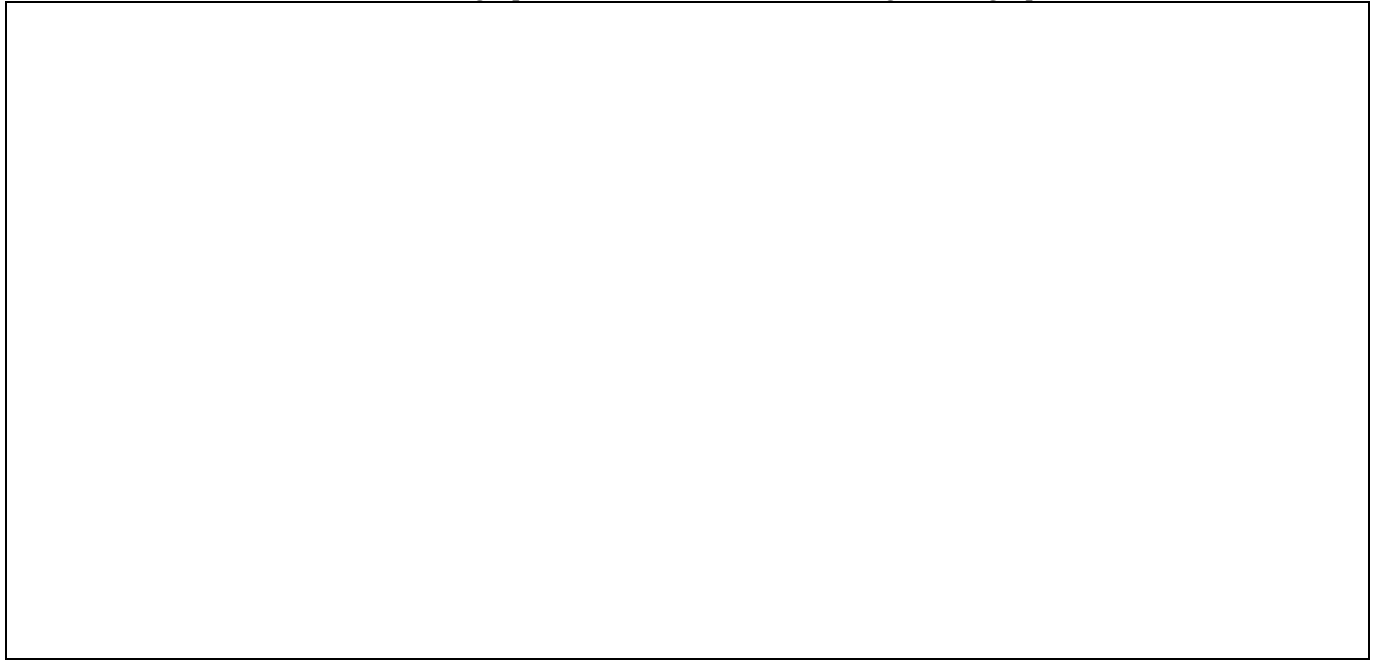


Figure 17.1: Plot of $|\text{sinc}(x)|$ function along with $1/x$.

In the case of general pulse width τ (ie, for arbitrary pulse train duty cycle $= \tau/T_1$), one should imagine the evenly spaced delta spikes superposed on the above picture. The spikes are located at $x_m = \omega_m\tau/2 = m\omega_1\tau/2 = m(\pi\tau/T_1)$, for $m = 1, 2, 3, \dots$. The spacing between the spikes is $dx = (\tau/T_1)\pi$. Thus, the number of spikes per hump is (T_1/τ) , since each hump is π wide. At low duty cycle, the spacing is small, and there are many lines for each "hump" of the $|\text{sinc}(x)|$ curve. Here is a rough plot for a ~4% duty cycle, $(T_1/\tau) = 22$:



Figure 17.2. Same $|\text{sinc}(x)|$ function with delta spike "lines". Height of each line is relative size of the c_m coefficient. Plot is for $\tau = T_1/22$, duty cycle about 4%

We shall now examine some special cases.

(a) If $\tau = T_1/2$ (50% duty cycle) we get a symmetric positive square wave pulse train, and (17.5) reduces to

$$c_m = (A/2) \text{sinc}(m\pi/2) \quad (17.7)$$

The DC component (the $m=0$ line) is $c_0 = (A/2)$, which is what we expect. All the other even lines vanish due to the sinc. For $m = \text{odd integers}$, we know that

$$\sin(m\pi/2) = (-1)^{(1-m)/2} = (i)^{1-m} = \text{real, since } m \text{ odd} \quad (17.8)$$

We summarize these facts for our symmetric pulse train with $t=0$ centered on a positive pulse:

$$\begin{aligned} c_m &= (A/\pi m)(i)^{1-m} & m &= \text{odd} \\ c_m &= 0 & m &= \text{even, } m \neq 0 \\ c_0 &= (A/2) \end{aligned} \quad (17.9)$$

In terms of Figure 17.1, the spacing between the spikes is $\pi/2$. Thus, all the $m = \text{even}$ spikes occur exactly at the zeros of the $\text{sinc}(x)$ function, that is why they all vanish. The $m=\text{odd}$ spikes occur centered between these zeros, very close to the peaks of the humps. The $1/m$ dropoff of the Fourier coefficients seen in (17.9) is reflected in our plot of $1/x$ in the picture. Ie, the $1/x$ curve intersects the odd spikes at the c_m coefficient values which are dropping off as $1/m$ (apart from overall constant).

Since all c_m in (17.9) are real, we know that $b_n = 0$ so there are only Fourier Series cosine contributions to $x(t)$.

What about that phase factor? The pulse train we have constructed above has $t=0$ occurring in the middle of a positive pulse. If we were to shift our entire pulse train to the left by $\tau/2$, so that a negative going pulse edge lines up with $t=0$, we would acquire an overall phase of $\exp(+i\omega\tau/2)$ which should then be applied to (17.4). At the lines $\omega = m \omega_1$ this phase becomes $\exp(i\pi m/2) = (i)^m$ acting on the c_m coefficients. This cancels the phase shown in (17.9) leaving only a constant phase of i .

So, here are the results for the same pulse train with $t=0$ occurring at a negative going edge:

$$\begin{aligned} c_m &= (iA/\pi m) & m &= \text{odd} \\ c_m &= 0 & m &= \text{even, } m \neq 0 \\ c_0 &= (A/2) \end{aligned} \quad (17.10)$$

Since all the odd- m c_m are now imaginary, we know that the corresponding a_m vanish, and only Fourier sines contribute to the above, as you would expect, since $x(t)$ is now an odd function of t .

(b) If $\tau = T_1$, we get from (17.5) that $c_m = A \text{sinc}(m\pi)$, so now *all* lines vanish except the line at $m=0$, which has a coefficient A . This is again reasonable, since such a pulse train is just a constant DC function $x(t) = A$. In terms of Figure 17.1, the pulse spacing is now π , and *all* the delta spikes align with zeros of the sinc function, except the DC line spike.

(c) If $\tau > T_1$, the theory still applies, but the waveforms are a bit strange looking since they overlap. As τ is continuously increased, the amount of overlap builds up, and the DC coefficient continues to increase, as shown in (17.6).

(d) If $\tau \rightarrow 0$, but $A\tau = \text{Area} = \text{held fixed}$, we have $x_{\text{pulse}}(t) = \text{Area} \delta(t)$. This is the limit of 0% duty cycle. Choosing $\text{Area} = 1$, we find that $X_{\text{pulse}}(\omega) = 1$, so that $c(\omega) = (1/T_1)$, just a constant:

$$\begin{aligned} x_{\text{pulse}}(t) &= \delta(t) & x(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_1) \\ X_{\text{pulse}}(\omega) &= 1 & X(\omega) &= \sum_{m=-\infty}^{\infty} (1/T_1) 2\pi \delta(\omega - m\omega_1) \end{aligned} \quad (17.11)$$

Thus, in the spectrum of a sequence of time-domain delta functions, all "lines" are present and have the same coefficient $(1/T_1)$. This function is often used as a sampling function in A/D conversion analysis. Notice that here, even though the time-domain pulse is a delta function, it's spectrum is still continuous - being a constant 1. The pulsetrain spectrum is discrete, as always. We shall have more to say later on the implications of (17.11).

(e) We have assumed that our time-domain pulse is centered at $t=0$. As noted earlier, if this is not the case, $X_{\text{pulse}}(\omega)$ picks up the phase $\exp(-i\omega a)$ where a is the new time origin of the pulse. Although this phase makes all the spectra complex, it does not affect any of our qualitative conclusions above, such as lines disappearing in certain cases. Also, the DC coefficients are unaffected since this $\exp(-i\omega a) = 1$ at $\omega=0$. As you slide the pulse train by varying point a , the mixture of real and imaginary part of the c_m varies. This corresponds to amplitude moving between the sine and cosine terms of the Fourier series.

18. Non-positive pulse trains.

This is pretty much a non-issue. We can take any pulse train described by coefficients c_m and superpose a constant DC level of say $-B$ units. This corresponds to $c_0 = -B$. Thus, if we choose $B = -A/2$, we can cancel out the DC level in our pulse train of (17.10).

So here are the results for a symmetric pulse train with no DC offset, with negative edge aligned on $t=0$, taken from (17.10) and cancelling the DC term.

$$\begin{aligned} c_m &= (iA/\pi m) & m &= \text{odd} \\ c_m &= 0 & m &= \text{even} \end{aligned} \quad (18.1)$$

19. Biphase pulse and pulsetrain.

Define a biphase pulse as being centered at $t=0$. The left pulse has width τ and amplitude $A/2$, the right pulse has width τ and amplitude $-A/2$, so the peak-to-peak amplitude is A , and a negative going edge aligns with $t=0$.

We can analyze this simply as the superposition of a positive and negative pulse of the square type studied above, but each pulse has amplitude $A/2$ instead of A . Also, the positive square pulse is time-shifted to the left by $\tau/2$ and the negative pulse is shifted to the right by $\tau/2$, so we pick up as corresponding spectral "shift phase" on each contributing pulse. The result is:

$$x_{\text{pulse}}(t) = \text{SquarePulse}(A/2, t+\tau/2) - \text{SquarePulse}(A/2, t - \tau/2) \quad (19.1)$$

$$\begin{aligned} X_{\text{pulse}}(\omega) &= ((A/2)\tau) \text{sinc}(\omega\tau/2) [e^{+i\omega\tau/2} - e^{-i\omega\tau/2}] \\ &= (A\tau) \text{sinc}(\omega\tau/2) [i \sin(\omega\tau/2)] \\ &= (2Ai/\omega) \sin^2(\omega\tau/2) \\ &= (iA\tau) \sin^2(x)/x \quad x = \omega\tau/2 \end{aligned} \quad (19.2)$$

This is the same as our envelope (9.2) for the positive pulse train (with pulse centered at $t=0$), except for the extra factor $[i \sin(\omega\tau/2)]$. Because of this extra factor, the coefficient envelope here is quite different than that for the square pulse. As $\omega \rightarrow 0$, the envelope function approaches zero -- there is no longer a

central hump. Here is a normalized plot comparing the positive pulse spectrum (9.2) to the biphasic spectrum (19.2):

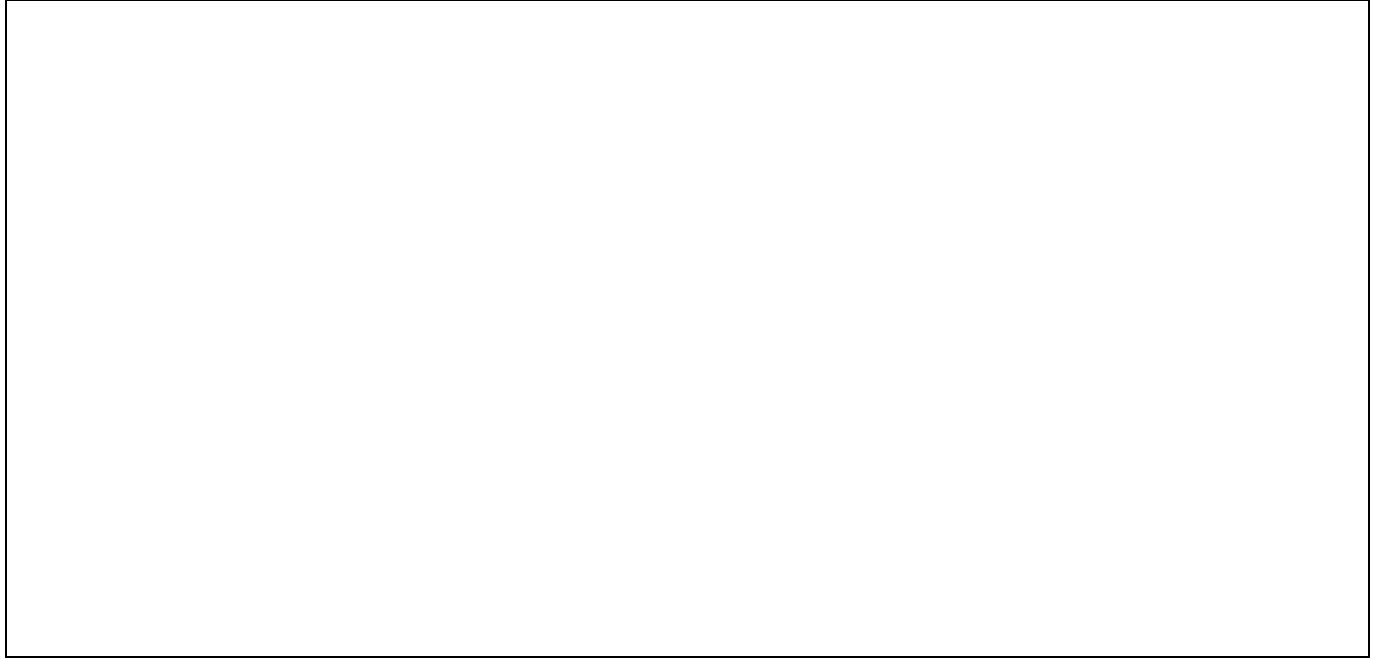


Figure 19.1. Same $|\text{sinc}(x)|$ and $1/x$ function, with biphasic $\sin^2(x)/x$ plot added.

Our biphasic pulsetrain is given by (17.4) and a new version of (17.5),

$$X(\omega) = \text{Error!}$$

$$c_m = (1/T_1) X_{\text{pulse}}(m\omega_1) = (iA\tau/T_1) \text{sinc}(m\pi\tau/T_1) \sin(m\pi\tau/T_1) \quad (19.3)$$

Notice that $c_0 = 0$ so that all biphasic pulse trains have zero DC offset.

Suppose now we select the special case $\tau = T_1/2$ in order to try and construct a symmetric square wave with zero DC component. The expression (19.3) reduces to:

$$c_m = (iA/m\pi) \sin^2(m\pi/2) \quad (19.4)$$

As before, the even lines all vanish. For the odd m lines, the phase factor in (17.8) is now squared, so it is always 1.

Thus, we summarize our results for a symmetric ($\tau = T_1/2$) biphasic pulse train:

$$\begin{aligned} c_m &= (iA/\pi m) & m &= \text{odd} \\ c_m &= 0 & m &= \text{even} \end{aligned} \quad (19.5)$$

As we hoped, this is identical to result (18.1) derived in a completely different manner.

To summarize, (19.5) is the spectrum of a symmetric square wave pulsetrain of period T_1 with no DC offset having a negative going edge align with $t=0$. The peak to peak amplitude is A . We derived (19.5) by superposing a set of time-shifted biphasic pulses.

The result agrees exactly with (18.1) which was obtained by a different process involving three steps: (1) treat a positive symmetric square wave with positive pulse centered at $t=0$; (2) shift it so that negative going edge aligns with $t=0$, thus changing the phase factor; (3) add a DC term to cancel the DC offset.

One might wonder how such different spectra envelopes (the two plots in the above figure) can yield exactly the same coefficients for $m = 1, 2, 3, \dots$ in the case $\tau = T_1/2$. The reason is easily understood. When $\tau = T_1/2$, the delta spikes in Figure 19.1 are positioned at $x = m\pi/2$, and at these points the two curves have the same values.

Here is a logarithmic view of the above figure. Of course $\log(0) = -\infty$, so these points are approximated on the graph. The log figure is interesting since it is what you see on the HP spectrum analyzer.



Figure 19.2. Logarithmic version of Figure 19.1.