

The Poisson sum formula

The Poisson sum formula takes on a number of different forms in the literature. Here is one useful version,

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m). \quad (1)$$

You will use this version of the Poisson sum formula in solving problem 14.13 of Jackson.

To prove this formula, consider the following periodic function, defined by:

$$f(x) = \frac{1}{2} - \frac{x}{2\pi}, \quad 0 \leq x \leq 2\pi, \quad (2)$$

where $f(x + 2\pi) = f(x)$. Note that $f(x)$ is discontinuous at $x = 2\pi m$, where $m = 0, \pm 1, \pm 2, \dots$. It follows that one can expand $f(x)$ in a Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x) dx.$$

Inserting $f(x)$ as given by eq. (2), one easily obtains:

$$c_0 = 0, \quad c_n = \frac{-i}{2\pi n}, \quad (n \neq 0).$$

That is,

$$f(x) = -\frac{i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{inx}}{n}. \quad (3)$$

Consider the derivative of $f(x)$, which we denote by $f'(x)$. From its definition [eq. (1)], $f'(x) = -1/(2\pi)$ for $x \neq 2\pi m$ ($m = 0, \pm 1, \pm 2, \dots$). At $x = 2\pi m$, the discontinuity of $f(x)$ can be described by the step function $\Theta(x)$. Specifically, in the vicinity of $x = 2\pi m$,

$$f(x) = -\frac{1}{2} + \Theta(x - 2\pi m), \quad \text{for } x \simeq 2\pi m. \quad (4)$$

That is, $f(x) = -1/2$ for $x = 2\pi m - \epsilon$ and $f(x) = 1/2$ for $x = 2\pi m + \epsilon$, where $\epsilon > 0$ is an infinitesimal quantity. Taking the derivative of eq. (4) yields:

$$f'(x) = \delta(x - 2\pi m), \quad \text{for } x \simeq 2\pi m.$$

We conclude that:

$$f'(x) = -\frac{1}{2\pi} + \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m). \quad (5)$$

We can also compute $f'(x)$ by differentiating the Fourier series of $f(x)$ term-by-term. Using eq. (3), we obtain:

$$f'(x) = \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{inx} = \frac{1}{2\pi} \left[-1 + \sum_{n=-\infty}^{\infty} e^{inx} \right]. \quad (6)$$

Equating eqs. (5) and (6) yields the desired result announced in eq. (1).

Actually, the most common form for the Poisson sum formula is as follows. Given a function $f(t)$ and its Fourier transform,

$$F(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt, \quad (7)$$

then the Poisson sum formula is given by:

$$\sum_{m=-\infty}^{\infty} f(\alpha m) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right), \quad (8)$$

where α is any number. One can derive eq. (8) by inserting the integral expression for F [eq. (7)] on the right-hand side of eq. (8), and then performing the sum over n using eq. (1). The resulting integrals are then trivially performed, and the left hand side of eq. (8) is immediately obtained.

For further details, see for example M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, pp. 67–71.