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ON THE THEORY OF DISPERSION OF X-RAYS

BY R. DE L. KRONIG

ABSTRACT

After a brief summary of the ideas underlying the quantum theory of dispersion it is shown that it can be applied to the refraction of x-rays, although the assumption that the number of atoms in a wave length cube is large is no longer satisfied. A general formula for the index of refraction in terms of the atomic absorption coefficient α and the critical frequencies is given. From the condition, experimentally verified, that the electrons in the atom for impressed frequencies, large compared to their natural frequencies, shall act like free electrons as far as the index of refraction is concerned, a relation is obtained for α . From the failure of this relation when applied to the groups of electrons separately, conclusions are drawn as to the coupling of the groups. Some considerations on the origin of the Compton shifted radiation are added, from which it appears that in the wave description this radiation must be regarded as coming from all the atoms and as being coherent with the incident waves; a result suited to stress the difficulty of harmonizing the wave picture with that of quantum processes in the atoms.¹

1. THE GENERAL DISPERSION THEORY

As is well known, a great many features of the phenomena arising from the action of external radiation upon matter in bulk can be qualitatively described by the classical theories. It is there assumed that all the atoms contain electric charges elastically and, in general, isotropically bound to equilibrium positions, having a damping proportional to their velocity. Under the influence of the radiation the resonators perform forced oscillations with the impressed frequency. On account of these oscillations they will emit spherical wavelets, their induced electric moment being parallel to, and almost in phase with, the electric field of the incident waves as long as the impressed frequency is sufficiently remote from any of the natural frequencies of the reson-

¹ As Professor Kramers kindly informs me, he too has obtained most of the results derived in 3. (To be published shortly.)

ators. These wavelets represent the scattered radiation, and by their interference with the primary waves they give rise to the retardation or acceleration of phase which causes the phenomenon of dispersion. When one of the natural frequencies of the resonators is approached by the impressed frequency, the amplitude of the forced oscillations becomes very large, and, moreover, they will be considerably out of phase with the electric field of the incident waves. The scattering increases markedly (resonance), and due to the phase shift, the wavelets produce a diminution of the amplitude of the incident wave (absorption). Finally, when the impressed frequency gets much greater than the natural frequency of a resonator, its charge will act as if free.

The induced electric moment of an atom containing charges $e_1, e_2, \dots, e_f$ of mass $m_1, m_2, \dots, m_f$ and capable of performing free oscillations about an equilibrium position with frequencies $\omega_1, \omega_2, \dots, \omega_f$ respectively is known to be

$$\mathfrak{M}(t) = \mathfrak{E} \sum_{i=1}^f \frac{e_i^2 \cos 2\pi\nu t}{4\pi^2 m_i (\omega_i^2 - \nu^2)} \quad (1)$$

if the impressed field at the point of the atom is given by

$$\mathfrak{E}(t) = \mathfrak{E} \cos 2\pi\nu t, \quad (2)$$

and ν is sufficiently different from the ω_i . The total amount of energy absorbed on the average by a resonator in a time dt from a radiation field, whose density $\rho(\nu)$ does not vary appreciably in the neighborhood of the natural frequency ω_i , is given by

$$\alpha_i c \rho(\omega_i) dt = \frac{\pi e_i^2}{m_i} \rho(\omega_i) dt,$$

where c is the velocity of light and α_i may be called the total atomic absorption coefficient for the resonator i .² Hence (1) may be written

$$\mathfrak{M}(t) = \mathfrak{E} \sum_i \frac{c \alpha_i \cos 2\pi\nu t}{4\pi^3 (\omega_i^2 - \nu^2)}. \quad (3)$$

The wave theory seems to be the only picture which affords an adequate description of the phenomenon of dispersion and its connection with absorption.³ Its ideas are therefore utilized in the quantum

² α_i is then the integral of the absorption coefficient over all frequencies, considering the absorption curve as approaching zero on both sides of the natural frequency. In reality the absorption coefficient at large frequencies approaches a small positive value corresponding to the scattering of a free charge. This part of the absorption coefficient has to be subtracted.

³ N. Bohr, *Zs. f. Phys.* 13, p. 161; 1923.

theory of dispersion by ascribing also here a scattering moment to every atom. Since, however, according to our knowledge of atomic structure, the picture of elastically and isotropically bound charges in the atom is not to be harmonized with other facts, Kramers⁴ investigated the perturbations arising in a multiply periodic system under the influence of an external field of the form (2) and found an expression for the induced scattering moment of an atom by introducing for the quantities characteristic of the multiply periodic system their quantum analogues. In the case of a large number of atoms in their normal state, oriented at random and having absorption lines at the frequencies ω_i , his general expression for the average scattering moment per atom reduces to⁵

$$\mathfrak{M}(t) = \mathfrak{E} \sum_i \frac{f_i e^2 \cos 2\pi \nu t}{4\pi^2 m(\omega_i^2 - \nu^2)}. \quad (4)$$

Here e and m are respectively the charge and mass of an electron, so that the atom acts as if containing resonators of charge $f_i e$ and mass $f_i m$. Kramers' expression for f_i may be written

$$f_i = \frac{mh\omega_i b_i}{\pi e^2},$$

where h is Planck's constant and b_i is defined by

$$P_i dt = b_i \rho(\omega_i) dt.$$

$P_i dt$ denotes the probability of an atom being lifted from the normal state to the state i in the time dt under the influence of a radiation field whose energy density between frequencies ω_i and $\omega_i + d\omega$ is $\rho(\omega_i) d\omega$. $b_i h \omega_i \rho(\omega_i) dt$ represents the energy absorbed on the average per atom in the time dt and may hence be put equal to $\alpha_i c \rho(\omega_i) dt$. This gives

$$f_i = \frac{mc\alpha_i}{\pi e^2}. \quad (5)$$

Introducing in (4) leads to the old expression (3) for the scattering moment of an atom. In fact, equation (4) was first derived by Laden-

⁴ H. A. Kramers, *Nature*, 113, p. 673; 114, p. 310, 1924; H. A. Kramers and W. Heisenberg, *Zs. f. Phys.*, 31, p. 681; 1925. See also M. Born, *Zs. f. Phys.*, 26, p. 379; 1924 and J. H. van Vleck, *Phys. Rev.*, 24, p. 344; 1924.

⁵ This is identical with a formula obtained previously by R. Ladenburg, *Zs. f. Phys.*, 4, p. 451; 1921 in a different way; see also R. Ladenburg and F. Reiche, *Naturwiss.*, 11, p. 584; 1923.

burg from the viewpoint of the very intimate connection between dispersion and absorption inherent in the wave picture.

2. CALCULATION OF THE INDEX OF REFRACTION

On account of what follows, it will be necessary to recall how the expression for the index of refraction can be derived from (4). Consider an infinite slab of the substance perpendicular to the direction of propagation Z of the incident waves and of thickness dz , small compared to the wave length λ , so that all the resonators inside will be practically in phase, say $dz = q\lambda$, where q is a small fraction. Take a point P a distance l away from the slab and put the Z -axis through it. The slab may be divided into circular zones by cylinders about Z of radius ρ . Let the breadth of such a zone be $d\rho$. We now investigate the resultant electromagnetic field at P produced by the superposition of the spherical wavelets originating from the atoms in the slab under the influence of a plane wave. For this purpose we must make the breadth of the zones such that waves coming from atoms on the inner and outer boundaries of the zone are still practically in phase; i.e., $dr = q\lambda$, where r is the distance from P to the zone. But $\rho d\rho = r dr = r q \lambda \geq l q \lambda$. Now the total volume of a zone is $2\pi \rho d\rho dz \geq 2\pi l q^2 \lambda^2$. But in order that one may be able, in computing the field at P , to integrate, it is necessary that in the volume $2\pi l q^2 \lambda^2$ there is a considerable number n of atoms. This gives the condition for l

$$2\pi l q^2 \lambda^2 N = n ,$$

$$l = \frac{n}{2\pi q^2 \lambda^2 N} , \quad (6)$$

where N is the number of atoms per unit volume.

Using for the scattering moment of an atom the expression (4), one easily finds by integrating over all the radiation fields of the oscillating moments that at distances greater than l the wavelets emitted by the slab under the influence of the incident wave have been ironed out by interference approximately into a plane wave. If we regard the refracting medium as made up of such slabs, we can neglect the effect on the motion of the resonators in a slab of the waves coming backwards from the following slabs, provided the polarization produced in the medium by the field E of the incident wave is small compared to E (i.e., $|\mu - 1| < < 1$). Then the retardation in phase of the incident wave in going

through a slab, produced by the wave from the slab, corresponds to an index of refraction μ given by

$$\mu - 1 = \delta = \sum_i \frac{Ne^2 f_i}{2\pi m(\omega_i^2 - \nu^2)}, \quad (7)$$

if all the atoms are of the same kind.

In the theory of dispersion in the optical region it is always assumed that there is a large number of atoms in a volume whose linear dimensions are of the same order of magnitude as the wave length, a condition generally satisfied in this region. The reason for the assumption is that, if the polarization of the substance is comparable to the electric field, then the only simple way of taking the effect of this polarization on an atom into account consists in imagining a sphere around the atom cut out of the substance. If the sphere can be so chosen that it contains many atoms, and yet its radius is small compared to the wave length, then the formulas for the polarization of a sphere in a uniform electric field can be applied. If, however, the polarization is very small so that it may be neglected, i.e., if μ does not differ much from unity, the preceding analysis shows that the above assumption no longer is necessary.⁶ Equation (7) then holds even if there are few atoms in a wave length cube, but it must be remembered that if actual refraction is to take place, the optical path of the beam under investigation must be greater than l .

3. APPLICATION TO THE DISPERSION OF X-RAYS

In case a substance composed of atoms in their normal state is illuminated by x-rays, we meet a problem with a number of interesting features. In the first place, there are very few atoms in a wave length cube, both the wave length and the distances between atoms being of the order of 1ÅU. Nevertheless, the formula given for the index of refraction will apply since $\mu - 1 = \delta$ is found to be of the order 10^{-6} , and l according to (6) ranges for solids around 10^{-3} cm if we assume as reasonable values for q and $nq = 1/25$, $n = 25$; while the actual path of the beam in a spectrometer is several cm. Further we have a continuous manifold of excited stationary states instead of a discrete sequence.

Consider an electron in one of the inner orbits of the atom, say an n_k orbit. Upon its removal the electron may have any velocity v .

⁶ For an elaborate mathematical discussion of the case of few atoms in a wavelength cube leading to the same conclusion the reader is referred to F. Reiche, *Ann. d. Phys.* 50, pp. 1 and 121, 1916.

The ionized atom will in general be capable of two states⁷ whose energy difference obeys the relativistic doublet formula, and which may be distinguished by a third quantum number j ; when $k=1$, the ionized atom is capable of one state only. Thus if an electron be removed from a 1_1 orbit, the atom remains in the state designated as K , while if an electron be removed from a 2_2 orbit, the atom may remain either in the state L_{II} or L_{III} , forming the relativistic L-doublet, these two states having the values $j=1$ and $j=2$ respectively.⁸ We may, therefore, characterize an excited state by (v, nkj) , or using as abbreviation for the combination nkj the letter r , we can denote it by (v, r) . Finally, we may call it (ω, r) , where ω is the frequency corresponding to a transition from the normal state to the state (v, r) . The continuous manifold of states (v, r) or (ω, r) , where r is given and v variable, corresponds to an absorption band extending from the critical frequency ω_r toward larger frequencies.

With a frequency interval $d\omega$ of the continuous absorption band r there is to be associated a resonator of charge $f_r(\omega)d\omega \cdot e$ and mass $f_r(\omega)d\omega \cdot m$, where $f_r(\omega)d\omega$ according to (5) is given by

$$f_r(\omega)d\omega = \frac{cm\alpha_r(\omega)d\omega}{\pi e^2}. \quad (8)$$

Here $\alpha_r(\omega)$ is the atomic absorption coefficient at the frequency ω due to transfer of atoms to the state r . In the expression (7) for the index of refraction we have now instead of a summation an integration by ω , extending over the absorption band r from ω_r to ∞ , and a summation over the various bands r and in addition terms corresponding to the lines associated with the transfer of outer electrons to higher stationary states. These latter each belong to a definite series limit at which an absorption band r begins and we may include them symbolically in the integration by extending it from 0 to ∞ . For a substance composed of atoms of one kind, one finds from (7) and (8)

$$\mu - 1 = \delta = \sum_r \int_0^\infty \frac{Nc\alpha_r(\omega)d\omega}{2\pi^2(\omega^2 - \nu^2)}. \quad (9)$$

This expression will hold even on the short wave length side of the absorption edges except in their immediate neighborhood, although then some of the resonators have frequencies nearly coinciding with ν .

⁷ Disregarding the different possible orientations of the orbits of the valence electrons.

⁸ These are the values customarily used in x-ray notation; their ratio is that of the statistical weights of the two states.

The reason is that the resonators in the narrow region from ν to $\nu + \Delta\omega$, in which the damping would become of importance, have their contribution almost canceled by the resonators from $\nu - \Delta\omega$ to ν . This is no longer true at the edge, where the damping must be taken into account. For compounds or mixtures the value of δ will be found additively from the values due to the components.

To proceed further it is necessary to know something about $\alpha_r(\omega)$. The general result of the investigations on the true absorption coefficients of elements of atomic number Z not too low and for wave lengths in the region from 0.1 ÅU to 1 ÅU appears to be⁹ that this coefficient is approximately expressible in the form

$$\alpha_r(\omega) = C_r Z^4 \lambda^3 = \frac{c^3 C_r Z^4}{\omega^3}, \quad (10)$$

C_r being a constant more or less independent of the substance. Such expressions have been found to hold both on the short and long wave length side of the K-absorption limit, i.e., for the absorption due to the K- and L-electrons. The experimental data¹⁰ on the refraction of x-rays have been obtained with frequencies ν large compared to the frequencies of most of the electrons in the atom. Since according to (10) $\alpha_r(\omega)$ decreases rapidly with ω , it is easily seen from (9) that the

⁹ For a detailed discussion of the experimental evidence and references to the literature see H. A. Kramers, *Phil. Mag.*, 46, p. 836; 1923; see also F. K. Richtmyer, *Phys. Rev.*, 27, p. 1; 1926.

¹⁰ Bergen Davis and H. M. Terrill, *Proc. Nat. Ac.*, 8, p. 537; 1922;
A. H. Compton, *Phil. Mag.*, 45, p. 1121; 1922;
C. C. Hatley and Bergen Davis, *Phys. Rev.*, 23, p. 290; 1924;
Bergen Davis and R. von Nardroff, *Phys. Rev.*, 23, p. 291; 1924; *Proc. Nat. Ac.*, 10, p. 60, 384; 1924;
C. C. Hatley, *Phys. Rev.*, 24, p. 486; 1924;
R. von Nardroff, *Phys. Rev.*, 24, p. 143; 1924;
E. Hjalmar and M. Siegbahn, *Nature*, 115, p. 85; 1925;
A. Larsson, M. Siegbahn and T. Waller, *Phys. Rev.*, 25, p. 235; 1925.
Bergen Davis and C. M. Slack, *Phys. Rev.*, 25, p. 881; 1925;
M. Siegbahn, *Journ. de Phys.*, 6, p. 228; 1925;
Bergen Davis and C. M. Slack, *Phys. Rev.*, 27, p. 18; 1926.
A. Larson, *Zs. f. Phys.* 35, p. 401; 1926;

I am indebted to Professor Davis and Mr. Slack for some additional unpublished material. This deals with the refraction of X-rays in silver on the long wavelength side of the K-limit. Below are given the values of $\delta \cdot 10^6$ as observed and as computed from the Lorentz formula and from (9) respectively: .707 Å.U.; $-5.3 \pm .6$, -5.7 , -6.0 ; .515 Å.U.; $-3.0 \pm .4$, -2.1 , -3.1 ; .500 Å.U., $-2.5 \pm .3$, -1.0 , -2.9 ; .485 Å.U., $-2.2 \pm .3$, 16.4, -2.5 .

contribution to δ of an absorption band due to such loosely bound electrons reduces very nearly to

$$-\frac{Nc}{2\pi^2\nu^2} \int_0^\infty \alpha_r(\omega) d\omega. \quad (11)$$

If ν is large compared to all the critical frequencies of the atom, then

$$\delta = -\frac{Nc}{2\pi^2\nu^2} \sum_r \int_0^\infty \alpha_r(\omega) d\omega.$$

Now the experimental values for δ in this case are found to check well with the value obtained by the classical theory on the assumption that all electrons act as if free:

$$\delta = -\frac{NZe^2}{2\pi m\nu^2},$$

where Z is the number of electrons per atom. From comparison there follows

$$\sum_r \int_0^\infty \alpha_r(\omega) d\omega = \frac{\pi Ze^2}{cm}.$$

This is a special case of a general condition formulated by Kuhn¹¹ and Thomas.¹² Kuhn, moreover, suggests from a consideration of the scattering activity of atoms at very high impressed frequencies that the sum of the charges $f_i e$ of the resonators associated with the absorption, which arises from transitions of electrons in a given kind of orbit of the normal state of the atom, is equal to the sum of the charges of these electrons. One thus would have from (11) in the case of the inner electrons which give rise only to continuous absorption

$$\sum_j \int_{\omega_{nkj}}^\infty \alpha_{nkj}(\omega) d\omega = \frac{\pi p_{nk} e^2}{cm}, \quad (12)$$

where p_{nk} is the number of electrons in the atom in its normal state occupying n_k orbits. For the K -electrons $\alpha_k(\omega)$ according to various observers⁹ is given by (10) with C_K about 0.02. This leads to a value

¹¹ W. Kuhn, *Zs. f. Phys.*, 33, p. 408; 1925.

¹² W. Thomas, *Naturwiss.*, 13, p. 627; 1925. See also F. Reiche and W. Thomas, *Zs. f. Phys.*, 34, p. 510; 1925.

for the above integral of about 0.025, while the right-hand side is about 0.05, p_K being equal to 2. The discrepancy between these values appears to indicate that the individual groups of electrons in the atom cannot be treated independently as regards the connection between their scattering activity at high frequencies and their absorbing properties.

The experimental data on refraction for an impressed frequency near one of the frequencies corresponding to an absorption edge are still too meager and too inaccurate to base any binding conclusions upon, especially so, since a large part of the refraction comes from the lightly bound electrons.

It may be remarked here that as regards the constants C_r in (10) corresponding to absorption edges which form a relativistic doublet some further information can be obtained from considerations similar to those applied by Heisenberg¹³ for an explanation of the summation rules in multiplets. For when an atom is transferred by removal of an inner electron from the normal state to either one of two higher states forming a relativistic doublet, the initial state is the same in both cases. The total radiative activity of the final states too may be expected to be the same, for in both the number of electrons in the different n_k orbits is the same. This leads by the familiar argument to the conclusion that the ratio of the constants C of the two absorption bands, disregarding quantities of the order $\Delta\omega/\omega$, is that of the statistical weights j of the final states, a result arrived at by Stoner¹⁴ in a different way.

4. THE NATURE OF THE COMPTON SCATTERED RADIATION

If the refracting medium is a light substance, and hard x-radiation be used, a considerable portion of the scattered radiation appears with its wave length shifted according to Compton's equation, the relative intensity of the shifted to the unshifted radiation increasing as the incident radiation is made harder, while the total amount of radiation is about what may be expected on the classical theory if all the electrons in the substance were free. This scattering, as is well known, is accompanied by $\bar{r}$ recoil electrons. The question arises whether the shifted radiation, in the wave picture, must be considered as coming from the electrons undergoing recoil, the electrons thus affected radiating very strongly, or whether all the atoms must be considered as participating equally in its production, radiating weakly. In actual experiments

¹³ W. Heisenberg, *Zs. f. Phys.*, 31 p. 617; 1925.

¹⁴ E. C. Stoner, *Phil. Mag.*, 48, p. 719; 1924.

carried out here on the index of refraction of carbon for molybdenum K -radiation, the intensity of the x-rays was at most of the order of magnitude 1 erg/cm²sec. As approximately 50 per cent of the scattered radiation is shifted, such an intensity would give rise to about 10^7 recoil electrons per cm³ per sec. Since the frequency of the incident radiation is about $4 \cdot 10^{18}$, and there will hardly be more than 10^6 waves in a wave train, the time during which an electron is engaged in a scattering process is not more than 10^{-12} sec. But if the shifted radiation came only from the electrons undergoing recoil, the above number of scattering processes would lead to the result that per cm³ at any time there are very few electrons or none at all engaged in this kind of scattering. The Compton shifted radiation could therefore not possibly contribute to the index of refraction by interference. Since, however, more and more of the scattered radiation becomes shifted radiation as the frequency increases, while at these high frequencies the index of refraction appears always to be what one would expect from the scattering of free electrons on the classical theory, it is evident that the shifted radiation, too, must in the wave picture be considered as coming from all the atoms, and that it must be coherent¹⁵ with the incident radiation. This is a case similar to that of the resonance radiation in atomic gases and vapors. There, too, the question arises if this resonance radiation, in the wave description, is to be considered as coming from all the atoms, being completely coherent with the incident light, or whether at least part of it comes from atoms in the upper stationary state, emitting strong radiation with no phase relation to the primary waves. From analogy one may be inclined to adopt the former view.

This state of affairs is particularly suited to accentuate the difficulties which one encounters in trying to connect the wave description with the appearance of the elementary processes, in our case the production of recoil electrons. These difficulties are all the greater since the experiments of Geiger and Bothe¹⁶ and of Compton and Simon¹⁷ have made

¹⁵ The shifted radiation can be considered, as far as the variation of its frequency with the angle of observation is concerned, as coming from sources moving in the direction of the incident beam with a velocity depending on the frequency of this beam (see A. H. Compton, *Phys. Rev.*, 21, p. 483; 1923). With coherence of the shifted radiation we mean that, if by a Lorentz transformation we go to a coordinate system in which the sources are at rest, then in this system the radiation coming from them has a definite phase relation to the incident waves. The fact that one does not observe selective reflection in crystals at angles corresponding to the shifted radiation does not contradict this, as the moving sources need not start moving all at the same time so that they no longer form a lattice.

¹⁶ W. Bothe and H. Geiger, *Zs. f. Phys.*, 32, p. 639; 1925.

¹⁷ A. H. Compton and A. W. Simon, *Phys. Rev.*, 26, p. 289; 1925.

it likely that giving up energy and momentum conservation in the individual processes, as suggested by Bohr, Kramers, and Slater,¹⁸ is not a feasible way out of the dilemma.

DEPARTMENT OF PHYSICS,
COLUMBIA UNIVERSITY,
NEW YORK CITY.
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¹⁸ Bohr, Kramers and Slater, *Phil. Mag.*, 47, p. 785, 1924; *Zs. f. Phys.*, 24, p. 69; 1924.

Instruments and Methods for locating Valuable Deposits.—

This is an article which summarizes various methods by which information may be gained about various kinds of deposits underlying a region of the earth's surface. The author discusses first the instruments by means of which the gravitational field of the region may be mapped. Among the instruments mentioned are Sterneck's pendulum apparatus in which four half-seconds pendulums are used; and the torsion balance of Eotvos and its application are discussed. The second general method consists of the mapping of the magnetic field of the region and suitable apparatus for this purpose is described. Measurements of temperature and of radioactivity at various points of the region being explored are mentioned as methods of obtaining useful information. The second group of methods are the acoustic method of Fessenden and the seismic of Mintrop. Another method which appears to have given dependable results is that of burying electrodes at suitable locations and sending currents of audible frequency through the earth by means of these electrodes. By means of "potential seekers" or probes the equipotential lines between the electrodes may be mapped. This method will reveal both conducting and insulating deposits. In conclusion the behavior of deposits toward electromagnetic waves (absorption, reflection, effect of dielectric constant) is mentioned as a factor of which advantage may be taken. The author gives a tabulated summary of the various methods, indicating in this summary the apparatus, the quantities measured, the variations produced in these quantities by deposits, and finally the kinds of deposits to which the methods are applicable. (C. Heiland, Berlin; *ZS. für Instrumentenkunde*, 45, pp. 417-436; 1925.)

PAUL E. KLOPSTEG.

The Measurement of Visibility with a Visibility Meter (Sichtmesser).—Estimation of visibility as modified by particles suspended in the air is usually accomplished in meteorology and navigation simply by observation with the unaided eye. In such an estimation the distance of visibility is given as the distance at which the observer is just able to distinguish an object. This method is convenient