This was installed into FT on Aug 8,2013

Appendix D: Calculation of a Sum which appears in (35.17)

We want to show that $(k = \omega T_1, z = e^{ik})$

$$S \equiv \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1} z^{-s} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)} \tag{D.1}$$

where the right side is $2\pi\delta_6(k,N)$ of (A.20). There are no doubt elegant ways to verify this identity, but we shall use the tried and true brute force method. We start off:

$$S = \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1} z^{-s} = \sum_{s=-2N}^{2N} z^{-s} - \frac{1}{2N+1} \sum_{s=-2N}^{2N} |s| z^{-s} = S_1 - \frac{1}{2N+1} S_2 \ . \tag{D.2}$$

The sum S_1 we know from (A.30) and (A.15) is

$$S_1 = \sum_{s=-2N}^{2N} z^{-s} = 2\pi \delta_5(k,2N) = \frac{\sin[(N+1/2)k]}{\sin(k/2)} \ . \tag{D.3}$$

So we work then on S_2 :

$$S_2 = \sum_{s=-2N}^{2N} |s| z^{-s} = \sum_{s=-2N}^{-1} |s| z^{-s} + \sum_{s=1}^{2N} |s| z^{-s} + 0$$

$$= \sum_{s=-2N}^{-1} (-s) z^{-s} + \sum_{s=1}^{2N} (s) z^{-s} = \sum_{s=1}^{2N} (s) z^{s} + \sum_{s=1}^{2N} (s) z^{-s} = \sum_{s=1}^{2N} s\,(z^s + z^{-s})$$

$$= 2 \sum_{s=1}^{2N} s \cos(ks) \ . \tag{D.4}$$

Suppose we define

$$S_3 \equiv \sum_{s=1}^{2N} \sin(ks) \ . \tag{D.5}$$

Then (here ∂_k means d/dk)

$$S_2 = 2\, \partial_k S_3. \tag{D.6}$$

So we now work on S_3 :

$$S_3 = \sum_{s=1}^{2N} \sin(ks) = (1/2i) \sum_{s=1}^{2N} [\, z^s - z^{-s}] = (1/2i) [\, \sum_{s=1}^{2N} z^s - \text{c.c.} \,]$$

$$= (1/2i) [\, S_4 - \text{c.c.} \,] \tag{D.7}$$

where c.c means complex conjugate. We next work on

$$S_4 = \sum_{s=1}^{2N} z^s = \sum_{s=0}^{2N} z^s - 1 \,. \tag{D.8}$$

Change summation variable to r = s-N so this becomes, again using (A.30),

$$S_4 = \sum_{r=-N}^{N} z^{(r+N)} - 1 = z^N \sum_{r=-N}^{N} z^r - 1 = z^N \, 2\pi \, \delta_5(k,N) - 1 \tag{D.9}$$

Then

$$[\, S_4 - \text{c.c.} \,] = \{ z^N \, 2\pi \, \delta_5(k,N) - 1\} - \{ z^{-N} \, 2\pi \, \delta_5(k,N) - 1\}$$
$$= (z^N - z^{-N}) \, 2\pi \, \delta_5(k,N) \,. \tag{D.10}$$

Backtracking now we find for (D.7) that

$$S_3 = (1/2i) [\, S_4 - \text{c.c.} \,] = (1/2i) \, (z^N - z^{-N}) \, 2\pi \, \delta_5(k,N) = \sin(Nk) \, 2\pi \, \delta_5(k,N) \,. \tag{D.11}$$

which gives a result we could have looked up (see Comment below),

$$\sum_{s=1}^{2N} \sin(ks) = \sin(Nk) \, \frac{\sin[(N+1/2)k]}{\sin(k/2)} \,. \tag{D.11}'$$

It remains to compute S_2 according to (D.6),

$$S_2 = 2 \, \partial_k \, S_3 = 2 \, \partial_k [\sin(Nk) \, 2\pi \, \delta_5(k,N)] \,. \tag{D.12}$$

Backing up more we then have from (D.2),

$$S = S_1 - \frac{1}{2N+1} \, S_2$$
$$= 2\pi \, \delta_5(k,2N) - \frac{1}{2N+1} \; 2 \, \partial_k [\sin(Nk) \, 2\pi \, \delta_5(k,N)] \tag{D.13}$$

so that

$$(2N+1)S = (2N+1) \, 2\pi \, \delta_5(k,2N) - 2\partial_k [\sin(Nk) \, 2\pi \, \delta_5(k,N)] \,. \tag{D.14}$$

Anticipating denominators of powers up to $\sin^2(k/2)$ we rewrite this as

$$\sin^2(k/2)\,(2N+1)S = (2N+1)\sin^2(k/2)[\,2\pi\,\delta_5(k,2N)] - 2\,\sin^2(k/2)\,\partial_k[\sin(Nk)\,2\pi\,\delta_5(k,N)]$$
$$= \quad T1 \quad - \quad T2$$

Then let

$$D5(k,N) \equiv [2\pi\,\delta_5(k,N)] = \frac{\sin[(N+1/2)k]}{\sin(k/2)} \qquad (A.15)$$

to get

$$T1 = (2N+1)\sin^2(k/2)\,D5(k,2N)$$

$$T2 = 2\sin^2(k/2)\,\partial_k[\sin(Nk)\,D5(k,N)]\,. \qquad (D.15)$$

The statement (D.1) which we are trying to prove is now this:

$$S = 2\pi\delta_6(n,K) = \frac{1}{2N+1}\,\frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)} \qquad (D.1)$$

and adding our factors to both sides this becomes

$$\sin^2(k/2)\,(2N+1)S = \sin^2(k/2)\,(2N+1)\;2\pi\delta_6(n,K)$$

$$= \sin^2(k/2)\,(2N+1)\,\{\frac{1}{2N+1}\,\frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)}\}$$

$$= \sin^2[(N+1/2)k]\,. \qquad (D.16).$$

If we then define this last factor as

$$T3 \equiv \sin^2[(N+1/2)k] \qquad (D.17)$$

<u>our task is then to show that</u>

$$T1 - T2 = T3$$

or

$$Q \equiv T1 - T2 - T3 = 0. \qquad (D.18)$$

where

$$T1 = (2N+1)\sin^2(k/2)\,D5(k,2N)$$
$$T2 = 2\sin^2(k/2)\,\partial_k[\sin(Nk)\,D5(k,N)]$$
$$D5(k,N) \equiv \frac{\sin[(N+1/2)k]}{\sin(k/2)}$$

$$T3 \equiv \sin^2[(N+1/2)k] \quad . \tag{D.19}$$

This is a task for Maple. We enter the quantities T1,T2,T3 and the function D5(k,N) :

```
T1 := (2*N+1)*sin(k/2)^2*D5(k,2*N);
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$$T1 := (2N+1)\sin\left(\frac{1}{2}k\right)^2 \mathrm{D5}(k, 2N)$$

```
T2 := 2*sin(k/2)^2*Diff(sin(N*k)*D5(k,N),k);
```

$$T2 := 2\sin\left(\frac{1}{2}k\right)^2 \left(\frac{\partial}{\partial k}\sin(Nk)\,\mathrm{D5}(k, N)\right)$$

```
D5 := (k,N) -> sin((N+1/2)*k)/sin(k/2);
```

$$\mathrm{D5} := (k, N) \to \frac{\sin\left(\left(N+\frac{1}{2}\right)k\right)}{\sin\left(\frac{1}{2}k\right)}$$

```
T3 := sin((N+1/2)*k)^2;
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$$T3 := \sin\left(\left(N+\frac{1}{2}\right)k\right)^2$$

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Q := T1-T2-T3;
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$$Q := (2N+1)\sin\left(\frac{1}{2}k\right)\sin\left(\left(2N+\frac{1}{2}\right)k\right) - 2\sin\left(\frac{1}{2}k\right)^2\left(\frac{\partial}{\partial k}\frac{\sin(Nk)\sin\left(\left(N+\frac{1}{2}\right)k\right)}{\sin\left(\frac{1}{2}k\right)}\right) - \sin\left(\left(N+\frac{1}{2}\right)k\right)^2$$

The Maple value command causes the differentiation to be carried out, so we continue :

```
:
> Q1 := simplify(value(Q));
```

$$Q1 := 2\sin\left(\frac{1}{2}k\right)\sin\left(\frac{1}{2}(4N+1)k\right)N + \sin\left(\frac{1}{2}k\right)\sin\left(\frac{1}{2}(4N+1)k\right)$$
$$- 2\cos(Nk)N\sin\left(\frac{1}{2}(2N+1)k\right)\sin\left(\frac{1}{2}k\right) - 2\sin(Nk)\cos\left(\frac{1}{2}(2N+1)k\right)\sin\left(\frac{1}{2}k\right)N$$
$$- \sin(Nk)\cos\left(\frac{1}{2}(2N+1)k\right)\sin\left(\frac{1}{2}k\right) + \sin(Nk)\sin\left(\frac{1}{2}(2N+1)k\right)\cos\left(\frac{1}{2}k\right) - 1 + \cos\left(\frac{1}{2}(2N+1)k\right)^2$$

```
> Q2 := expand(Q1);
```

$$Q2 := 2\sin\left(\frac{1}{2}k\right)^2 N\cos(Nk)^2 - 2\sin\left(\frac{1}{2}k\right)^2 N + 2\sin\left(\frac{1}{2}k\right)^2\cos(Nk)^2 - \sin\left(\frac{1}{2}k\right)^2$$
$$+ 2\sin(Nk)^2\sin\left(\frac{1}{2}k\right)^2 N + 2\sin(Nk)^2\sin\left(\frac{1}{2}k\right)^2 + \sin(Nk)^2\cos\left(\frac{1}{2}k\right)^2 - 1 + \cos(Nk)^2\cos\left(\frac{1}{2}k\right)^2$$

```
> Q3 := simplify(Q2);
```

$$Q3 := 0$$

Thus we have shown that Q = 0 so (D.1) is then verified. **QED**

Comment: Gradshteyn and Ryzhik provide the following summation formulas :

1.342

1. $$\sum_{k=1}^{n} \sin kx = \sin \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$$

2.[10] $$\sum_{k=0}^{n} \cos kx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1$$
$$= \cos \frac{nx}{2} \sin \frac{n+1}{2} x \operatorname{cosec} \frac{x}{2} = \frac{1}{2} \left(1 + \frac{\sin \left(n + \frac{1}{2} \right) x}{\sin \frac{x}{2}} \right)$$

1.352

1.[11] $$\sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \left(\frac{2n-1}{2} x \right)}{2 \sin \frac{x}{2}}$$

2.[11] $$\sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \left(\frac{2n-1}{2} x \right)}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}$$

An alternate method of verifying (D.1) would be to use two of these sums in an approach that begins this way

$$S \equiv \sum_{s=-2N}^{2N} \frac{2N+1 - |s|}{2N+1} z^{-s} = \sum_{s=-2N}^{2N} \frac{2N+1 - |s|}{2N+1} (z^{-s} + z^{s})/2 = \sum_{s=-2N}^{2N} \frac{2N+1 - |s|}{2N+1} \cos(ks)$$

$$= 1 + 2 \sum_{s=1}^{2N} \frac{2N+1 - s}{2N+1} \cos(ks) = 1 + 2 \sum_{s=1}^{2N} \cos(ks) - 2 \sum_{s=1}^{2N} s \cos(ks)$$