Appendix F retired Aug 8, 2013

Appendix F: The Spectrum and Power Density for Repeated-Sequence Pulse Trains

Overview: For infinite pulse trains composed of repeats of some length-P subsequence:

(a) computes X(ω) in (F.12)
(b) computes $\mathcal{P}(\omega)$ in (F.23)
(c) computes $<\mathcal{P}(\omega)>$ for an ensemble of pulse trains which respect the special condition

$<y_m^* y_n> = \alpha$ for $m \neq n$

$<y_m^* y_n> = \beta$ for $m = n$ $\quad s < |m-n|$ *(F.25)*

The result for $<\mathcal{P}(\omega)>$ is stated in (F.33) in several different forms.
(d) takes the $P \rightarrow \infty$ limit of the section (c) result for $<\mathcal{P}(\omega)>$
(e) computes $\mathcal{P}(\omega)$ for a single pulse train which respects the special condition

$<y_m y_n>_1 = \alpha$ for $m \neq n + NP$ $\quad N$ = any integer

$<y_m y_n>_1 = \beta$ for $m = n + NP$ *(F.43)*

where $<y_m y_n>_1$ is a horizontal average across the single sequence (autocorrelation).
The result for $\mathcal{P}(\omega)$ is stated in (F.52).
It is noted that the results for $<\mathcal{P}(\omega)>$ of (c) and $\mathcal{P}(\omega)$ of (e) are exactly the same in terms of their respectively defined α and β constants.
(f) restates the results of (c) and (e) as Fact 1 (F.53) and Fact 2 (F.54).
A graphical representation is drawn for the spectrum in general, and for a box pulse.
It is shown that the MLS sequence is a candidate for application of Fact 2. A simple ensemble is then constructed to which Fact 1 maybe applied, and the result is then interpreted.
(g) treats the P = 2 repeated subsequence A,B using the general formulas of (a) and (b)

We start with this collection of equations:

$$Y''(z) = \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \qquad \text{Z Transform of } y_n \qquad (24.2) \qquad \text{(F.1)}$$

$$|Y''(z)|^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad \text{(F.2)}$$

$$X(\omega) = X_{pulse}(\omega) Y''(z) \qquad (25.3) \qquad \text{(F.3)}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} |Y''(z)|^2 \qquad (34.14) \qquad \text{(F.4)}$$

$$z = e^{i\omega T_1} \qquad (24.1)$$

$$\omega_1 \equiv 2\pi/T_1$$

In (F.4) T is the duration of the infinite pulse train, as in (33.22). The pulse train amplitudes are the y_n. Since sequence y_m is composed of subsequences of length P that repeat, we have this periodicity property of the y_n

$$y_{m+IP} = y_m \qquad \text{for any integer I} \tag{F.5}$$

(a) Calculation of X(ω) for a Pulse Train with a Repeated Sequence

Consider first the sum in (F.1). Let n = IP + n' and write this sum as

$$\sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} = \sum_{I=-\infty}^{\infty} \sum_{n'=0}^{P-1} y_{IP+n'} e^{-i\omega (n'+IP) T_1} = \sum_{I=-\infty}^{\infty} e^{-i\omega IP T_1} \sum_{n'=0}^{P-1} y_{n'} e^{-i\omega n' T_1}$$

$$= \left(\sum_{I=-\infty}^{\infty} e^{-i\omega IP T_1} \right) \left(\sum_{n=0}^{P-1} y_n e^{-i\omega n T_1} \right) . \tag{F.6}$$

We see that the sum factors into the product of two sums. The first sum we evaluate using

$$\sum_{n=-\infty}^{\infty} e^{-ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \quad . \qquad -\infty < k < \infty \tag{A.31}$$

Setting $k = \omega P T_1$ we find

$$\left(\sum_{I=-\infty}^{\infty} e^{-i\omega IP T_1} \right) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P T_1 - 2\pi m) \quad . \tag{F.7}$$

The second sum we give the name $Y_P''(z)$ which is the Z transform of the subsequence $\{ y_0, y_1 ..... y_{P-1} \}$.

$$Y_P''(z) \equiv \sum_{n=0}^{P-1} y_n e^{-i\omega n T_1} \quad . \tag{F.8}$$

Thus we have shown that

$$Y''(z) = \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} = Y_P''(z) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P T_1 - 2\pi m) \quad , \tag{F.9}$$

and then from (F.3).

$$X(\omega) = X_{pulse}(\omega)\, Y''(z) = X_{pulse}(\omega)\, Y_P''(z) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P T_1 - 2\pi m) . \tag{F.10}$$

It is convenient to write

$$2\pi\delta(\omega PT_1 - 2\pi m) = 2\pi(PT_1)^{-1}\delta(\omega - 2\pi m/PT_1) = (1/P)\,\omega_1\,\delta(\omega - m\omega_1/P) \qquad (F.11)$$

and then

$$X(\omega) = X_{pulse}(\omega)\,\omega_1\,(1/P)\,Y_P''(z) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \qquad (F.12)$$

$$\text{where} \quad Y_P''(z) \equiv \sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1} \quad . \qquad (F.8)$$

X(ω) is the Fourier Transform Spectrum of an infinite pulse train composed of a repeating P-length subsequence. Since the pulse train is periodic, the spectrum is entirely discrete with lines at

$$\omega_m = (m/P)\omega_1. \qquad (F.13)$$

(b) Calculation of $\mathcal{P}(\omega)$ for a Pulse Train with a Repeated Sequence

In Section 35 [see below (35.26)'] it was noted that $\mathcal{P}(\omega)$ can be calculated either by the Autocorrelation approach or the Double Sum approach. Here we shall take the latter approach based on (F.2) above.

We can reorganize the double sum in (F.2) into a quadruple sum by defining :

$$n = IP + n'$$
$$m = JP + m' .$$

Then the double sum above becomes,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{JP+m'})^* (y_{IP+n'})\, e^{i\omega P(I-J)T_1}\, e^{i\omega(m'-n')T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{m'})^* (y_{n'})\, e^{i\omega P(I-J)T_1}\, e^{i\omega(m'-n')T_1} \quad , \qquad (F.14)$$

where in the last line we have used the periodicity (F.5) of the y_n. The following illustration shows how, in this reorganization, we first sum over a square grid patch with n' and m', and then we sum over an array of those patches with I and J.

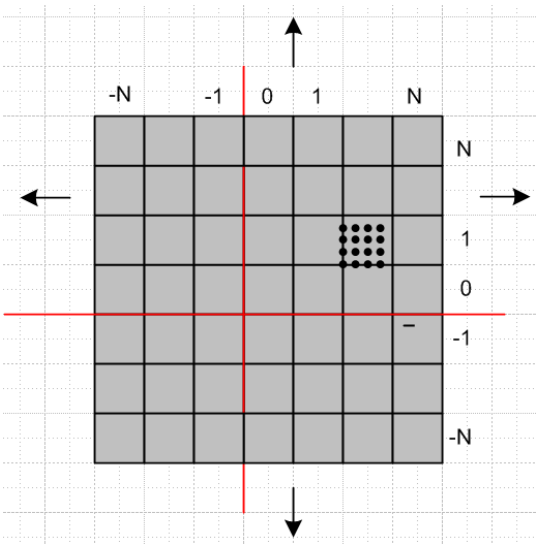


Fig F.1

In order to regulate things, we shall assume that the I and J sums range from -N to N rather than from -∞ to ∞. This means we are assuming that the sequence is (2N+1) repeated periods in length and not infinite. Then of course we can say

$$T = (2N+1)PT_1 \,. \tag{F.15}$$

As usual, we keep N finite as long as possible, and then take N→ ∞ in the end.

We now rewrite (F.2) by removing the primes from summation indices and reordering the factors

$$|\, Y''(z)\,|^2 \;=\; \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \;\rightarrow$$

$$= \Big[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \Big] \Big[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \Big]$$

$$= \Big[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \Big]\, |\, Y_P''(z)\,|^2 \tag{F.16}$$

As happened in (F.6) with a single sum, our double sum factors into a product of two double sums. The second double sum we recognize from (F.8) as $|\, Y_P''(z)\,|^2$. The first double sum is

$$\Big[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \Big] = \Big|\, \sum_{I=-N}^{N} e^{i\omega PIT_1} \,\Big|^2 \tag{F.17}$$

Then apply (A.30)

$$\sum_{n=-N}^{N} e^{ink} = 2\pi\,\delta_5(k,N) = 2\pi\,\{\ \frac{\sin[(N+1/2)k]}{2\pi\sin(k/2)}\ \} \qquad (A.30)$$

with $k = \omega P T_1$ to get

$$[\ \sum_{I=-N}^{N}\ \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}\ ] = |\ 2\pi\,\delta_5(\omega P T_1,N)\ |^2 = (\ 2\pi\,\delta_5(\omega P T_1,N)\ )^2\ . \qquad (F.18)$$

Then from (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi}\ \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1}\ \frac{\sin^2[(N+1/2)k]}{2\pi\sin^2(k/2)}\ , \qquad (A.20)$$

we can write the first factor of (F.16) as

$$[\ \sum_{I=-N}^{N}\ \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}\ ] = (2N+1)\,2\pi\,\delta_6(\omega P T_1,N) = (1/P)\,\frac{T}{T_1}\,2\pi\,\delta_6(\omega P T_1,N) \qquad (F.19)$$

where T is from (F.15). At this point we have, looking at (F.16) and (F.19),

$$|\ Y''(z)\ |^2 = [\ \sum_{I=-N}^{N}\ \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}\ ]\ [\ \sum_{n=0}^{P-1}\ \sum_{m=0}^{P-1} y_m^*\ y_n\ e^{i\omega(m-n)T_1}\ ]$$

$$= [\ (1/P)\,\frac{T}{T_1}\,2\pi\,\delta_6(\omega P T_1,N)\ ]\ \ |\ Y_P''(z)\ |^2\ . \qquad (F.20)$$

Now finally we take $N\to\infty$ using (A.21)

$$\lim_{N\to\infty}\delta_6(k,N) = \sum_{m=-\infty}^{\infty}\delta(k-2\pi m) \qquad (A.21)$$

or

$$\lim_{N\to\infty}\delta_6(\omega P T_1,N) = \sum_{m=-\infty}^{\infty}\delta(\omega P T_1-2\pi m) \qquad (F.21)$$

with this result

$$|\ Y''(z)\ |^2 = [\ (1/P)\,\frac{T}{T_1}\,2\pi\sum_{m=-\infty}^{\infty}\delta(\omega P T_1-2\pi m)\ ]\ \ |\ Y_P''(z)\ |^2$$

or

$$\frac{T_1}{T}\ |\ Y''(z)\ | = (1/P)\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1-2\pi m)\ |\ Y_P''(z)\ |^2$$

Using (F.11) we can rewrite this as

$$\frac{T_1}{T} \; | \, Y''(z) \, | = \omega_1 \, (1/P)^2 \sum_{m = -\infty}^{\infty} \delta(\omega - m\omega_1/P) \; | \, Y_P''(z) \, |^2 \, . \qquad \text{(F.22)}$$

Installing this into (F.4) then gives

$$\mathcal{P}(\omega) \; = \mathcal{P}_{pulse}(\omega) \; \omega_1 \, (1/P)^2 \sum_{m = -\infty}^{\infty} \delta(\omega - m\omega_1/P) \; | \, Y_P''(z) \, |^2 \qquad \text{(F.23)}$$

$$\text{where} \quad Y_P''(z) \equiv \sum_{n = 0}^{P-1} y_n \, e^{-i\omega n T_1} \quad . \qquad \textit{(F.8)}$$

$\mathcal{P}(\omega)$ is the Spectral Power Density of an infinite pulse train composed of a repeating P-length subsequence. The power density is discrete with lines at $\omega_m = (m/P)\omega_1$, the same lines observed in the spectrum $X(\omega)$ of (F.12).

(c) Calculation for an Ensemble of such Pulse Trains subject to Certain Conditions

We now imagine an ensemble of P-length subsequences s_i. We then create a corresponding ensemble of infinite length sequences S_i according to $S_i = \{.....s_i, s_i, s_i, s_i, \; ...\}$. This just an arbitrary ensemble, not a random ensemble or any other special kind ensemble.

Then we apply the ensemble average <..> to (F.23) to get

$$\langle \mathcal{P}(\omega) \rangle \; = \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m = -\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \sum_{n = 0}^{P-1} \sum_{m = 0}^{P-1} \langle y_m^* \, y_n \rangle \; e^{i\omega(m-n)T_1} \qquad \text{(F.24)}$$

At this point, *suppose it happens* that

$$\langle y_m^* \, y_n \rangle \; = \alpha \qquad \text{for } m \neq n$$

$$\langle y_m^* \, y_n \rangle \; = \beta \qquad \text{for } m = n \qquad \text{(F.25)}$$

where α and β are *independent* of the indices shown. Notice in the (F.24) sum that max(n-m) = P-1, so we don't have to worry about these indices differing by an integral multiple of P. We have now restricted our interest to the sequence $\{y_0, y_1, \ldots y_{P-1}\}$.

Comment: In the case of a random ensemble, (F.25) would be true if the random variables Y_m and Y_n associated with pulse train locations m and n were independent, but saying that α and β are independent of the indices covers a larger class of possibilities, including some in which Y_m and Y_n are in fact correlated, hence dependent. In other words, saying that α and β as shown above don't depend on the indices m and n does not imply that $\langle y_m^* \, y_n \rangle = \langle y_m^* \rangle \langle y_n \rangle$ which is the definition of being independent.

Then we can write

$$\sum_{n=0}^{P-1}\sum_{m=0}^{P-1} \langle y_m^* y_n \rangle e^{i\omega(m-n)T_1} = \alpha \sum_{n=0}^{P-1}\sum_{m\neq n} e^{i\omega(m-n)T_1} + \beta \sum_{n=0}^{P-1} 1$$

$$= \alpha \sum_{n=0}^{P-1}\sum_{m\neq n} e^{i\omega(m-n)T_1} + \beta P \ . \qquad \text{(F.26)}$$

To evaluate the double sum, we write it as

$$\sum_{n=0}^{P-1}\sum_{m\neq n} e^{i\omega(m-n)T_1} = \sum_{n=0}^{P-1}\sum_{m=0}^{P-1} e^{i\omega(m-n)T_1} - \sum_{n=0}^{P-1}\sum_{m=n} e^{i\omega(m-n)T_1}$$

$$= \left| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \right|^2 - \sum_{n=0}^{P-1} 1 = \left| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \right|^2 - P \ . \qquad \text{(F.27)}$$

Now let $\omega T_1 = k$ and consider

$$\sum_{n=0}^{P-1} e^{-ink} = \sum_{n=0}^{P-1} (e^{-ik})^n = \sum_{n=0}^{P-1} x^n = 1 + x + \ldots + x^{P-1} = \frac{1-x^P}{1-x}$$

$$= \frac{1-e^{-ikP}}{1-e^{-ik}} \ . \qquad \text{(F.28)}$$

With a Maple assist,

```
assume(k,real): assume(P,real);
f := (1-exp(-I*k*P))/(1-exp(-I*k));
```

$$f := \frac{1 - e^{(-I k P)}}{1 - e^{(-I k)}}$$

```
f1 := abs(f^2);
```

$$f1 := \frac{(1-\cos(kP))^2 + \sin(kP)^2}{(1-\cos(k))^2 + \sin(k)^2}$$

```
f2 := simplify(f1);
```

$$f2 := \frac{-1+\cos(kP)}{-1+\cos(k)}$$

```
f3 := subs(cos(k) = 1 - 2*sin(k/2)^2,f2);
```

$$f3 := -\frac{1}{2}\frac{-1+\cos(kP)}{\sin\left(\frac{1}{2}k\right)^2}$$

```
f4 := subs(cos(k*P) = 1 - 2*sin(k*P/2)^2,f3);
```

$$f4 := \frac{\sin\left(\frac{1}{2}kP\right)^2}{\sin\left(\frac{1}{2}k\right)^2}$$

we see that

$$\left| \sum_{n=0}^{P-1} e^{-ink} \right|^2 = \left| \frac{1-e^{-ikP}}{1-e^{-ik}} \right|^2 = \frac{\sin^2(kP/2)}{\sin^2(k/2)} \quad . \tag{F.29}$$

But from (A.20)

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \quad . \tag{A.20}$$

setting P = 2N+1 we recognize $\frac{\sin^2(kP/2)}{\sin^2(k/2)}$ as one of our multiple delta function models, so then

$$\left| \sum_{n=0}^{P-1} e^{-ink} \right|^2 = \left| \frac{1-e^{-ikP}}{1-e^{-ik}} \right|^2 = \frac{\sin^2(kP/2)}{\sin^2(k/2)} = 2\pi P \, \delta_6(k, \frac{P-1}{2}) \quad . \tag{F.30}$$

If we set $k = \omega T_1$, we can insert (F.30) into (F.27) with this result

$$\sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} = \left| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \right|^2 - P = 2\pi P \, \delta_6(\omega T_1, \frac{P-1}{2}) - P$$

$$= P \, [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1] \quad . \tag{F.31}$$

We now go back to (F.24)

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \tag{F.13}$$

and replace the double sum using (F.26) to get

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \, [\alpha \, \{ \sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} \} + \beta P \,] \, . \tag{F.32}$$

Then we use (F.31) for {...} to get

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \, [\alpha \, \{ \, P \, [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1] \, \} + \beta P \,]$$

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P} \, [\alpha \, \{ \, [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1] \, \} + \beta \,]$$

or

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P}\ [(\beta-\alpha)\ + \alpha\, 2\pi\, \delta_6(\omega T_1, \frac{P-1}{2})\ ] \qquad \text{(F.33a)}$$

where

$$\delta_6(k,N) \equiv \frac{1}{2N+1}\ \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \qquad \Rightarrow \qquad 2\pi\, \delta_6(\omega T_1, \frac{P-1}{2}) = \frac{1}{P}\ \frac{\sin^2(P\omega T_1/2)}{\sin^2(\omega T_1/2)}\ .$$

Evaluating at the delta function hit values $\omega = \omega_1 m/P$ we find

$$2\pi\, \delta_6 \ = \frac{1}{P}\frac{\sin^2(m\pi)}{\sin^2(m\pi/P)} \ = \begin{cases} 0 & m \neq NP \\ P & m = NP \end{cases} \quad \text{for N = any integer}\ .$$

To get alternate forms for (F.33a), we first express each term as a separate sum,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1\ \frac{1}{P}\ [(\beta-\alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\, 2\pi\delta_6(\omega T_1, \frac{P-1}{2})\ ]\ .$$

Since only those m which are multiples of P contribute to the second sum in (F.33a), we may rewrite that second sum as follows,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1\ \frac{1}{P}\ [(\beta-\alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{N=-\infty}^{\infty} \delta(\omega - \omega_1 N)\, P\ ] \qquad \text{// m = NP}$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1\ \ [(\beta-\alpha)\ \frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.33b)}$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\, \delta(\omega - \omega_1 m/P)\ +\ \alpha\, \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.33c)}$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1\ [\ (\beta-\alpha)\ \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta-\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]\ . \qquad \text{(F.33d)}$$

If $x_{\mathtt{pulse}}(t)$ is real, then by (7.5) $\mathcal{P}_{\mathtt{pulse}}(\omega)$ is an even function of ω, and we can then reflect the negative part of the sum to the positive side to get,

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1\ [\ (\beta-\alpha)\ \frac{2}{P} \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ 2\alpha \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta-\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]\ . \qquad \text{(F.33e)}$$

In all forms of (F.33) we have: $\alpha = \langle y_m^*\, y_n \rangle$ for $m \neq n$ $\beta = \langle y_n^*\, y_n \rangle$.

These slightly different forms of $\langle \mathcal{P}(\omega) \rangle$ are useful for different purposes. All results are valid for any finite integer P. Since each sequence in the ensemble is periodic with the same period P, the ensemble average spectrum is entirely discrete. The m≠0 sums include positive and negative integers.

(d) Limit as P → ∞ of the Ensemble Result

We would now like to take the limit of the above as P→∞. Write (F.33b) as

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [(\beta-\alpha)\ \{\frac{\omega_1}{P}\sum_{m=-\infty}^{\infty}\delta(\omega-\omega_1 m/P)\}\ +\ \omega_1\ \alpha\ \sum_{m=-\infty}^{\infty}\delta(\omega-\omega_1 m)\ ]\quad . \tag{F.34}$$

Then define

$$f_P(\omega) \equiv \frac{\omega_1}{P}\sum_{m=-\infty}^{\infty}\delta(\omega-\omega_1 m/P)\ . \tag{F.35}$$

As P→∞, the spacing of the δ lines becomes closer and closer, while the amplitude $\frac{\omega_1}{P}$ of each δ line becomes less and less. Perhaps we can argue that in the limit this becomes some continuous function.

In line with the distribution theory approach to symbolic functions noted in Appendix A, suppose we integrate this function from some a to a+ε for small ε, where a is an arbitrary real number,

$$\int_a^{a+\varepsilon} f_P(\omega)\,d\omega = \frac{\omega_1}{P}\sum_{m=-\infty}^{\infty}\int_a^{a+\varepsilon}\delta(\omega-\omega_1 m/P)$$

$$= \frac{\omega_1}{P}\sum_{m=-\infty}^{\infty}\Theta(a \le \omega_1 m/P \le a+\varepsilon) \tag{F.36}$$

where we use the notation of Appendix A (e) for the Θ function which takes value 1 if the inequality argument is valid, meaning there is a delta hit. If P is a large integer, how many non-zero terms does this Σ_m have? The inequality argument reads

$$Pa/\omega_1 \le m \le Pa/\omega_1 + P\varepsilon/\omega_1\ .$$

Since P is large, we round each term in this equation to the nearest integer, making little error. We select a very small ε first, and then we make sure P is large enough so $P\varepsilon/\omega_1$ is still a reasonably large integer when rounded. Then we have

$$\int_a^{a+\varepsilon} f_P(\omega)\,d\omega = \frac{\omega_1}{P}\sum_{m=Pa/\omega_1}^{Pa/\omega_1+P\varepsilon/\omega_1}\Theta(a \le \omega_1 m/P \le a+\varepsilon) = \frac{\omega_1}{P}\sum_{m=Pa/\omega_1}^{Pa/\omega_1+P\varepsilon/\omega_1} 1$$

$$= \frac{\omega_1}{P}\ (P\varepsilon/\omega_1) = \varepsilon\ . \tag{F.37}$$

Since we then have (for very large P) that $\int_a^{a+\varepsilon} f_P(\omega)\,d\omega = \varepsilon$ for any real a and for ε as small as we like, and since the integral over range ε is proportional to ε, the function $f_P(\omega)$ is equivalent to the constant function 1. Thus we have shown that,

$$\lim_{P\to\infty} f_P(\omega) = \lim_{P\to\infty} \left[\frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \right] = 1. \qquad \text{(F.38)}$$

We then obtain this P→∞ limit of (F.34),

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \left[(\beta-\alpha) + \omega_1 \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m) \right] \qquad \text{(F.39)}$$

$$= \mathcal{P}_{pulse}(\omega) \left[(\beta-\alpha) + (1/T_1) \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_1 m) \right] \qquad // \ \omega_1 = 2\pi/T_1$$

$$= \mathcal{P}_{pulse}(\omega) \left[(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right] .$$

This limit agrees with our result (35.11) for a random ensemble of infinite sequences for which $\langle a_m a_n \rangle$ does not depend on the values of m and n,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \qquad \textit{(35.11)}$$

In the limit P→∞ , the discrete lines $\delta(\omega - \omega_1 m/P)$ shown in (F.34) have coalesced into a continuous function. See Fig F.2 below for a graphical view.

(e) Calculation of $\mathcal{P}(\omega)$ for a single P-periodic Pulse Train subject to Certain Conditions

Recall this expression for $\mathcal{P}(\omega)$, which incorporates the discrete Wiener-Khintchine relation,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} |\, Y''(z) \,|^2 = \mathcal{P}_{pulse}(\omega)\, R''(z) \qquad \textit{(34.14a)} \qquad \text{(F.40)}$$

where R"(z) is the Z transform of the autocorrelation sequence r_s obtained from the y_n.

Our first task is to find r_s, which we defined this way,

$$r_s \equiv \lim_{N\to\infty} \left[\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s} \right] = \langle y_n\, y_{n+s} \rangle_1 . \qquad \textit{(32.16)} \qquad \text{(F.41a)}$$

Since y_n is periodic as shown in (F.5), r_s may be written in this alternate form:

$$r_s = \frac{1}{P} \sum_{n=0}^{P-1} y_n\, y_{n+s} = \langle y_n\, y_{n+s} \rangle_1 \qquad \text{(F.41b)}$$

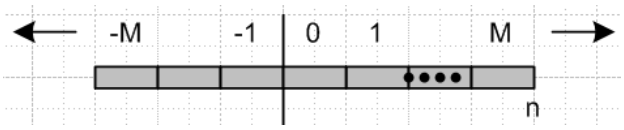
Proof: For large N, we can replace the sum endpoints by -N = -MP and +N = MP ≈ (M+1)P where M is also large. The sum then has (2M+1)P terms, so

$$r_s \equiv \lim_{M\to\infty} \left[\frac{1}{(2M+1)}\frac{1}{P}\sum_{n=-MP}^{(M+1)P} y_n\, y_{n+s}\right].$$

We now let n = IP + n' where n' takes values 0,1..P-1 and I = Int(n/P). Then the single sum above can be written,

$$\sum_{n=-MP}^{(M+1)P} y_n\, y_{n+s} = \sum_{I=-M}^{M}\sum_{n'=0}^{P-1} y_{IP+n'}\, y_{IP+n'+s}$$

where this drawing shows how the sum is now a double sum where I denotes segments containing P points, and n' counts the points in each segment,



We then process this sum using the periodicity (F.5) to get

$$= \sum_{I=-M}^{M}\sum_{n'=0}^{P-1} y_{n'}\, y_{n'+s} = \left(\sum_{I=-M}^{M} 1\right)\left(\sum_{n=0}^{P-1} y_n\, y_{n+s}\right) = (2M+1)\left(\sum_{n=0}^{P-1} y_n\, y_{n+s}\right).$$

Inserting the expression into r_s, we get this alternate form for r_s

$$r_s = \lim_{M\to\infty}\left[\frac{1}{(2M+1)}\frac{1}{P}\left\{(2M+1)\left(\sum_{n=0}^{P-1} y_n\, y_{n+s}\right)\right\}\right] = = \frac{1}{P}\sum_{n=0}^{P-1} y_n\, y_{n+s} \qquad \textbf{QED}$$

From now on we assume that the y_m are real. Back in (F.25) we made the assumption that

$$\langle y_m y_n\rangle = \alpha \qquad \text{for } m \neq n \qquad s < |\, m-n\,|$$
$$\langle y_m y_n\rangle = \beta \qquad \text{for } m = n \qquad \textit{(F.25)}$$

where <...> were ensemble averages. Here we are going to make a completely different assumption, and this assumption applies to a single sequence:

$$\langle y_m y_n\rangle_1 = \alpha \qquad \text{for } m \neq n \qquad s < |\, m-n\,|$$
$$\langle y_m y_n\rangle_1 = \beta \qquad \text{for } m = n \qquad \text{(F.42)}$$

Comment: If we imagine drawing each sequence of an ensemble as a row in a set of rows

... * * y_n * y_{n+2} ... sequence #1

...	*	*	y'_n	*	y'_{n+2}	...	sequence #2
...	*	*	y''_n	*	y''_{n+2}	...	sequence #3

then any ensemble average like $\langle y_n y_{n+2} \rangle$ is a **vertical average** through the ensemble, whereas an average like $\langle y_n y_{n+2} \rangle_1$ is a **horizontal average** across one particular sequence row and $\langle y_n y_{n+2} \rangle_1$ never depends on n as (F.41) shows. These two averages are unrelated and (F.42) being true does not imply that (F.25) is true. As an example, one might take 10 infinite sequences each of which satisfies (F.42) and put them down as a set of rows, and then each row is shifted horizontally some amount to cause a 0 to be in column n. For the resulting 10 row ensemble, one would find that $\langle y_n^2 \rangle = 0$ for column n, whereas $\langle y_n^2 \rangle_1 = \beta$.

Since we require r_s for arbitrary s in order to compute R"(z), we extend (F.42) in this manner

$$\begin{aligned} &\langle y_m y_n \rangle_1 = \alpha \qquad \text{for } m \neq n + NP \qquad N = \text{any integer} \\ &\langle y_m y_n \rangle_1 = \beta \qquad \text{for } m = n + NP \end{aligned} \tag{F.43}$$

which is to say, for s being any integer whatsoever,

$$\begin{aligned} &\langle y_n\, y_{n+s} \rangle_1 = \alpha \qquad s \neq NP \\ &\langle y_n\, y_{n+s} \rangle_1 = \beta \qquad s = NP \quad . \end{aligned} \tag{F.44}$$

The reason of course is that $\{y_m\}$ is periodic with period P,

$$y_{m+NP} = y_m \qquad \text{for any integer N} \tag{F.5}$$

so we must have for example $\langle y_n\, y_{n+P} \rangle_1 = \langle y_n\, y_n \rangle_1 = \beta$.

Thus we have arrived at our characterization of the autocorrelation sequence r_s for our specific infinite periodic sequence with the assumption (F.44),

$$r_s = \langle y_n\, y_{n+s} \rangle_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer} \tag{F.45}$$

Our next step is to compute its Z transform R"(z),

$$\begin{aligned} R''(z) &= \sum_{n=-\infty}^{\infty} r_n z^{-n} = \beta \sum_{n=NP} z^{-n} + \alpha \sum_{n \neq NP} z^{-n} \\ &= \beta \sum_{n=NP} z^{-n} + \alpha \left(\sum_{n=-\infty}^{\infty} z^{-n} - \sum_{n=NP} z^{-n} \right) \\ &= (\beta - \alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n} \quad . \end{aligned} \tag{F.46}$$

The first sum is over n = 0, ±P, ±2P and so on. We can replace summation index n by index N,

$$\sum_{n=NP} z^{-n} = \sum_{N=-\infty}^{\infty} z^{-NP} = \sum_{n=-\infty}^{\infty} z^{-nP} \quad . \tag{F.47}$$

From (24.1) we know that z lies on the unit circle in the z-plane and is related to ω by

$$z = e^{i\omega T_1} \tag{24.1}$$

where T_1 is the duration of a pulse of the pulse train. We then have

$$\sum_{n=NP} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-nP} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 nP} . \tag{F.48}$$

According to (A.31),

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \tag{A.31}$$

so setting $k = -\omega T_1 P$ we get

$$\sum_{n=NP} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(-\omega T_1 P - 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P + 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) \tag{F.49}$$

where we use $\delta(x) = \delta(-x)$ and in the last step take $m \rightarrow -m$.

Meanwhile, our other sum of interest in (F.46) is this one,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-n} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 n}$$

which is just the previous sum without the P. Thus,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \tag{F.50}$$

Inserting (F.48) and (F.49) into (F.46) gives

$$R''(z) = (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n}$$

$$= (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (\beta-\alpha)\,(T_1 P)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/[T_1 P]) + \alpha (T_1)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/T_1)$$

$$= (2\pi/T_1)\ \{\ (\beta-\alpha)\,(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/[T_1 P]) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/T_1)\ \}$$

$$= \omega_1\ \{\ (\beta-\alpha)\,(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\ \} \qquad \text{(F.51)}$$

and this concludes our calculation of the Z Transform R"(z) of the autocorrelation sequence r_s. It only remains to install this into the Z Transform Wiener-Khintchine relation (34.14a) which says

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R"(z) \qquad \textit{(34.14a)}$$

so then

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \omega_1\ \{\ (\beta-\alpha)\,\frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\ \}$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\,\delta(\omega - \omega_1 m/P)\ +\ \alpha\,\delta(\omega - \omega_1 m)\ ] \qquad \text{(F.52c)}$$

We may compare this to the ensemble result of the section (c) above,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\,\delta(\omega - \omega_1 m/P)\ +\ \alpha\,\delta(\omega - \omega_1 m)\ ] \quad . \qquad \textit{(F.33c)}$$

The expressions are exactly the same, but the meaning of symbols α and β is not the same! Since the expressions have the exact same form, we can rewrite (F.52c) in the same alternate ways that (F.33c) was written:

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P}\ [(\beta-\alpha)\ +\ \alpha\, 2\pi\, \delta_6(\omega T_1, \frac{P-1}{2})\ ] \qquad \text{(F.52a)}$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1\ \frac{1}{P}\ [(\beta-\alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{N=-\infty}^{\infty} \delta(\omega - \omega_1 N)\ P\ ] \qquad \text{// } m = NP$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1\ \ [(\beta-\alpha)\ \frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.52b)}$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\,\delta(\omega - \omega_1 m/P)\ +\ \alpha\,\delta(\omega - \omega_1 m)\ ] \qquad \text{(F.52c)}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) \ + \ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) \ + \{ (\beta-\alpha)/P + \alpha \} \delta(\omega) \right] \tag{F.52d}$$

(f) Summary of Results of (c) and (e) and an Example: The MLS Sequence

It is useful to summarize both results more completely:

Fact 1: (for an ensemble of infinite periodic sequences) If (F.53)

$$\begin{aligned} &\langle y_m y_n \rangle = \alpha \qquad \text{for } m \neq n \qquad\qquad s < |\, m-n \,| \\ &\langle y_m y_n \rangle = \beta \qquad \text{for } m = n \end{aligned} \tag{F.25}$$

then

$$\langle \mathcal{P}(\omega) \rangle \ = \ \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \left[(\beta-\alpha) \ \frac{1}{P} \delta(\omega - \omega_1 m/P) \ + \ \alpha \, \delta(\omega - \omega_1 m) \ \right] \tag{F.33c}$$

Fact 2: (for a single infinite periodic sequence) If (F.54)

$$\begin{aligned} &\langle y_m y_n \rangle_1 = \alpha \qquad \text{for } m \neq n + NP \qquad\qquad N = \text{any integer} \\ &\langle y_m y_n \rangle_1 = \beta \qquad \text{for } m = n + NP \end{aligned} \tag{F.43}$$

then

$$\mathcal{P}(\omega) \ = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \left[(\beta-\alpha) \ \frac{1}{P} \delta(\omega - \omega_1 m/P) \ + \ \alpha \, \delta(\omega - \omega_1 m) \ \right] \tag{F.52c}$$

In Fact 2, the parameters α and β can be interpreted as elements of the autocorrelation seqeunce

$$r_s = \langle y_n \, y_{n+s} \rangle_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer} \tag{F.45}$$

Here are the two results (F.33d) and below that (F.52d), where as noted both expressions are the same,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) \ + \ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) \ + \{ (\beta-\alpha)/P + \alpha \} \delta(\omega) \right]$$

$$\mathcal{P}(\omega) \ = \ \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) \ + \ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) \ + \{ (\beta-\alpha)/P + \alpha \} \delta(\omega) \right]$$

Then here is a graphical representation the above spectral power densities :

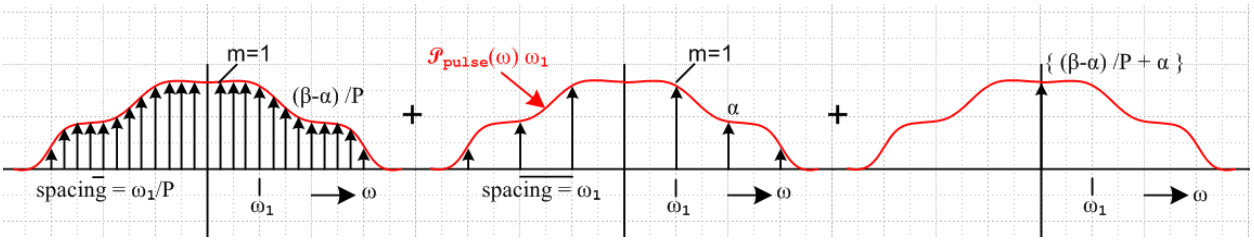
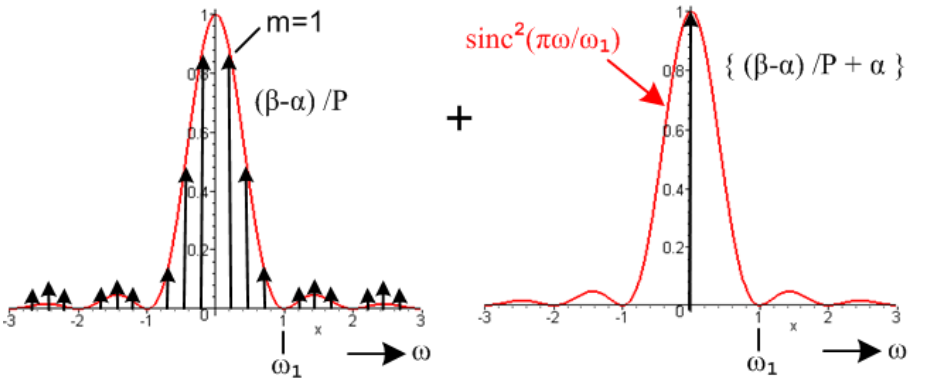


Fig F.2

Each vertical arrow represents a spectral δ line. The height of the arrow is the value of the red envelope curve times the quantity shown. The red curve $\mathscr{P}_{pulse}(\omega)\ \omega_1$ will in general have an infinite extent, an example being $\mathscr{P}_{pulse}(\omega)\ \omega_1 = \text{sinc}^2(\omega T_1/2) = \text{sinc}^2(\pi\omega/\omega_1)$ for a box shaped pulse as in (9.2). Once a particular $\mathscr{P}_{pulse}(\omega)$ is specified, some of the spectral lines may be quenched. For the box example, lines are quenched when $\omega = N\omega_1$ for $N = \pm1, \pm2$ .... In this case, all the lines of the central image go away and the corresponding lines in the left image also vanish, this being every P^{th} line in that image:



for box pulse shape

Fig F.3

As P → ∞, we showed in section (d) that either of our two spectra above become

$$\mathscr{P}_{pulse}(\omega)\ \omega_1\ [\ (\beta-\alpha)\ +\ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m)\ +\ \alpha\ \delta(\omega)\ ]$$

Regarding the three images of Fig F.2, we see that

- the left set of dense arrows coalesces into the continuous function $(\beta-\alpha)\ \mathscr{P}_{pulse}(\omega)\ \omega_1$
- the middle set of arrows stays exactly the same
- the amplitude of the DC line becomes α

Example: The MLS Sequence

A Maximum Length Sequence (MLS) (Lucht *Polynomial Multipliers...*) which is created by a shift register generator has the property (F.44), specifically,

$$\langle y_n\ y_{n+s} \rangle_1 = \alpha = (1/4)(1 + 1/P) \qquad\qquad s \neq NP$$

$$\langle y_n\, y_{n+s}\rangle_1 = \beta = (1/2)(1 + 1/P) \qquad s = NP \qquad \textit{(F.44)}$$

where P must be one of the special values $P = 2^k\text{-}1$ for k = 1,2,3.... Therefore, by Fact 2 (F.54) the spectrum $\mathcal{P}(\omega)$ of an MLS sequence is given by

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\,\delta(\omega - \omega_1 m/P) + \alpha\,\delta(\omega - \omega_1 m)\ ]\ . \qquad \textit{(F.52)}$$

The horizontal average $\langle a_n\, a_{n+s}\rangle_1$ of a sequence $\{a_i\}$ is seen in this picture made for P = 7 and s = 2 :

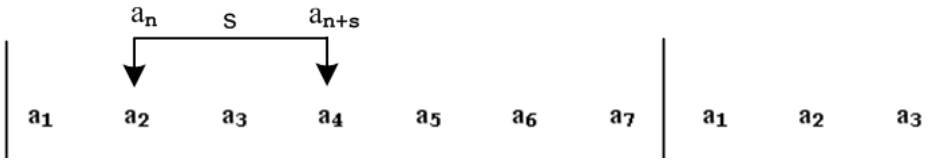


Fig F.4

The double-arrow pointer points to a_2 and a_4 and their product is a_2a_4. The pointer is rigidly slid horizontally so the left arrow points to a_1 through a_7 in turn. The sum of the pointed-to products is added up and the result divided by P, and that is $\langle a_n\, a_{n+s}\rangle_1$.

Now we can find a role for our ensemble Fact 1 as follows. Suppose we make a special ensemble of P sequences in which each is shifted one bit to the left of the one above. Then consider this picture of the entire ensemble (clipped horizontally of course since each sequence is infinite),

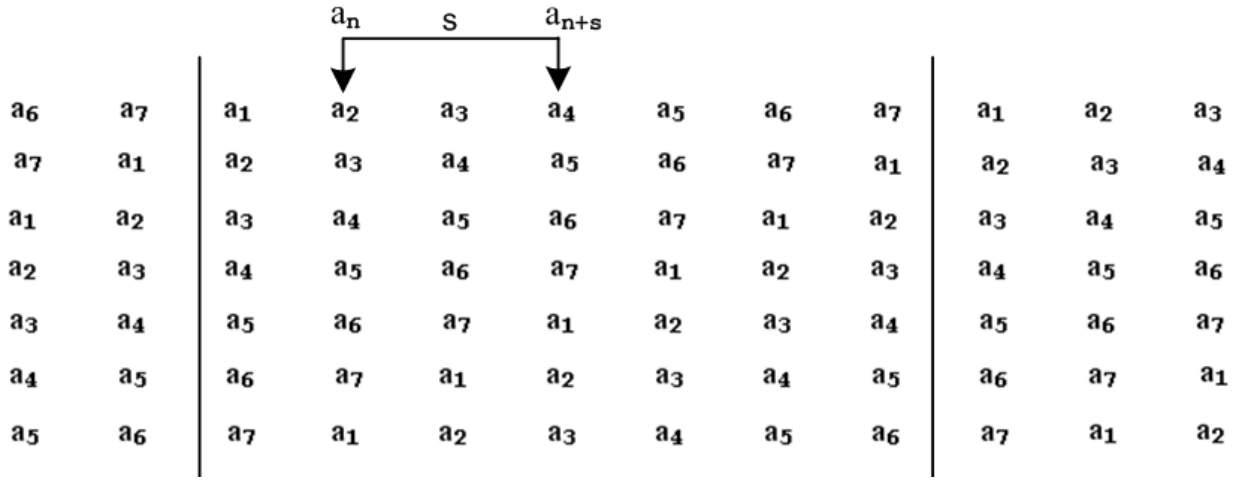


Fig F.5

We now leave the double-arrow pointer horizontally fixed and apply it vertically to each row one at a time. For the first row we get the product a_2a_4 and for the second row a_3a_5. We add up the products and divide by P. This process generates the vertical average $\langle a_n\, a_{n+s}\rangle$. But clearly $\langle a_n\, a_{n+s}\rangle = \langle a_n\, a_{n+s}\rangle_1$ for this special ensemble since the sums are exactly the same. Thus, (F.43) implies (F.25) with the same α and β and we can then apply Fact 1 (F.53) to find

$$\langle \mathcal{P}(\omega) \rangle \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega)\;\; \omega_1 \sum_{m\,=\,-\infty}^{\infty} \left[\, (\beta\text{-}\alpha)\;\frac{1}{P}\,\delta(\omega - \omega_1 m/P) \;+\; \alpha\,\delta(\omega - \omega_1 m)\;\right]$$

Comparing our results, we have for this ensemble that $\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$ of any sequence in the ensemble. this says that $\mathcal{P}(\omega)$ is the same for every sequence. This conclusion is in agreement with the fact that when an infinite sequence is shifted some number of bit positions, its spectral power density cannot change.

(g) Results for an A,B repeated sequence

This subject is treated in Section 34 (c) using a "brute force" approach and a "Fourier Series" approach. Here we duplicate the main results of that section by applying our general formulas to a sequence where the repeated subsequence is just P = 2, $\{y_0,y_1\} = \{A,B\}$, and we allow A and B to be complex. The fact that the results here agree with Section 34 lends some confidence to all three methods of computation.

Our general expressions for X(ω) and $\mathcal{P}(\omega)$ are,

$$X(\omega) \;= X_{\mathtt{pulse}}(\omega)\;\omega_1\;(1/P)\;Y_{\mathtt{P}}''(z) \sum_{m\,=\,-\infty}^{\infty} \delta(\omega - m\omega_1/P) \qquad \textit{(F.12)}$$

$$\mathcal{P}(\omega) \;= \mathcal{P}_{\mathtt{pulse}}(\omega)\;\omega_1\;(1/P)^2 \sum_{m\,=\,-\infty}^{\infty} \delta(\omega - m\omega_1/P)\;\; |\, Y_{\mathtt{P}}''(z)\,|^2 \qquad \textit{(F.23)}$$

$$\text{where} \quad Y_{\mathtt{P}}''(z) \;\equiv \sum_{n\,=\,0}^{P-1} y_{\mathtt{n}}\, e^{-i\omega n T_1} \quad . \qquad \textit{(F.8)}$$

For a repeated A,B sequence P = 2 we have

$$Y_{\mathtt{P}}''(z) \;= A + B\, e^{-i\omega T_1}$$

$$|\, Y_{\mathtt{P}}''(z)\,|^2 = |\, A + B\, e^{-i\omega T_1}\,|^2 \;= |A|^2 + |B|^2 + 2\,\mathrm{Re}\{A^* B\, e^{-i\omega T_1}\}$$

Then,

$$X(\omega)_{\mathtt{AB}} \;= X_{\mathtt{pulse}}(\omega)\;\omega_1\;(1/2)\;\left[A + B\, e^{-i\omega T_1}\right] \sum_{m\,=\,-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{\mathtt{AB}} \;= \mathcal{P}_{\mathtt{pulse}}(\omega)\;\omega_1\;\;(1/4)\,[\;\; |A|^2 + |B|^2 + 2\,\mathrm{Re}\{A^* B\, e^{-i\omega T_1}\}\;] \sum_{m\,=\,-\infty}^{\infty} \delta(\omega - m\omega_1/2) \;\; .$$

We can slide the square-bracketed factors inside the sum and then use

$$\omega T_1 = (m\omega_1/2)T_1 = \pi\, m(\omega_1/2\pi)T_1 = \pi m \qquad \Rightarrow \qquad e^{-i\omega T_1} \;=\; (-1)^{\mathtt{m}}$$

and the results simplify to

$$X(\omega)_{AB} = X_{pulse}(\omega)\,\omega_1\,(1/2)\sum_{m=-\infty}^{\infty}[A + B(-1)^m]\,\delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{AB} = \mathcal{P}_{pulse}(\omega)\,\omega_1\,(1/4)\sum_{m=-\infty}^{\infty}[\,|A|^2 + |B|^2 + 2\,\mathrm{Re}(A^*B)(-1)^m\,]\,\delta(\omega - m\omega_1/2)\quad .$$

These results agree with (34.18) and (34.20), see just below Fig 34.1 .

For a **box pulse** of height 1 and width T_1 we know that

$$X_{pulse}(\omega) = T_1\,\mathrm{sinc}(\omega T_1/2) = (1/\omega_1)\,2\pi\,\mathrm{sinc}(\omega T_1/2) \qquad (9.2)$$

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)\ . \qquad (33.24)$$

Then for a square wave with alternating A,B amplitudes we get

$$X(\omega)_{AB} = 2\pi\,\mathrm{sinc}(\omega T_1/2)\,(1/2)\sum_{m=-\infty}^{\infty}[A + B(-1)^m]\,\delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{AB} = \mathrm{sinc}^2(\omega T_1/2)\,(1/4)\sum_{m=-\infty}^{\infty}[\,|A|^2 + |B|^2 + 2\,\mathrm{Re}(A^*B)(-1)^m\,]\,\delta(\omega - m\omega_1/2)\quad .$$

When the sinc functions are moved inside the sum, $\omega T_1/2 = \pi m/2$, so

$$X(\omega)_{AB} = 2\pi\,(1/2)\sum_{m=-\infty}^{\infty}[A + B(-1)^m]\,\mathrm{sinc}(\pi m/2)\,\delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{AB} = (1/4)\sum_{m=-\infty}^{\infty}[\,|A|^2 + |B|^2 + 2\,\mathrm{Re}(A^*B)(-1)^m\,]\,\mathrm{sinc}^2(\pi m/2)\delta(\omega - m\omega_1/2)$$

But

$$\text{sinc}(\pi m/2) = (2/\pi m)\sin(\pi m/2) \quad = 1 \text{ for } m = 0$$
$$= 0 \text{ for } m \text{ even}$$
$$= (2/\pi m)(-1)^{(m-1)/2} \text{ for } m \text{ odd}$$

so we get

$$X(\omega)_{AB} = 2\pi (1/2) \sum_{m=\text{odd}} [A - B(-1)^m](2/\pi m)(-1)^{(m-1)/2}\,\delta(\omega - m\omega_1/2) + 2\pi(1/2)[A+B]\,\delta(\omega)$$

$$\mathcal{P}(\omega)_{AB} = (1/4) \sum_{m=\text{odd}} [\,|A|^2 + |B|^2 + 2\,\text{Re}(A^*B)(-1)^m\,](2/\pi m)^2(-1)^{(m-1)}\,\delta(\omega - m\omega_1/2)$$
$$+ \omega_1\,(1/4)[\,|A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,]\,\delta(\omega)$$

or

$$X(\omega)_{AB} = 2 \sum_{m=\text{odd}} [A - B](1/m)(-1)^{(m-1)/2}\,\delta(\omega - m\omega_1/2) + \pi[A+B]\,\delta(\omega)$$

$$\mathcal{P}(\omega)_{AB} = (1/\pi^2) \sum_{m=\text{odd}} [\,|A|^2 + |B|^2 - 2\,\text{Re}(A^*B)\,](1/m)^2\,\delta(\omega - m\omega_1/2)$$
$$+ \omega_1\,(1/4)[\,|A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,]\,\delta(\omega)$$

For a standard-issue square wave with B = - A we have

$$[A - B] = 2A \qquad [\,|A|^2 + |B|^2 - 2\,\text{Re}(A^*B)\,] = 4|A|^2$$
$$[A + B] = 0 \qquad [\,|A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,] = 0$$

and therefore

$$X(\omega)_{A,-A} = 4A \sum_{m=\text{odd}} (1/m)(-1)^{(m-1)/2}\,\delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{A,-A} = 4A^2\,(1/\pi^2) \sum_{m=\text{odd}} (1/m)^2\,\delta(\omega - m\omega_1/2)$$

in agreement with (34.23).