This was the original version of Appendix F, it got replaced by the v2 file.

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Appendix F: The Power Density of an Infinite Pulse Train having a Repeated Sequence

(a) The Spectral Power Density for a Single Pulse Train with a Repeated Sequence

The problem here is to write $\mathcal{P}(\omega)$ for an infinite pulse train whose coefficients are periodic with period P. The result is stated below in (F.12). Then in the ensemble special case that $\langle a_m a_n \rangle$ is independent of the index values m and n, a result for $\langle \mathcal{P}(\omega) \rangle$ is given in (F.22). The limit P→∞ of that equation is taken with the result shown in (F.28).

We start with our expression (34.14) for the spectral power density of an infinite sequence,

$$\mathcal{P}(\omega) = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \qquad (34.14)$$

where T is the duration of the infinite sequence (to which we return soon). Suppose the sequence y_m is composed of subsequences of length P that repeat. Then

$$y_{m+IP} = y_m \qquad \text{for any integer I} \qquad (F.1)$$

We can reorganize the double sum in (34.14) into a quadruple sum by defining :

$$n = IP + n'$$
$$m = JP + m' .$$

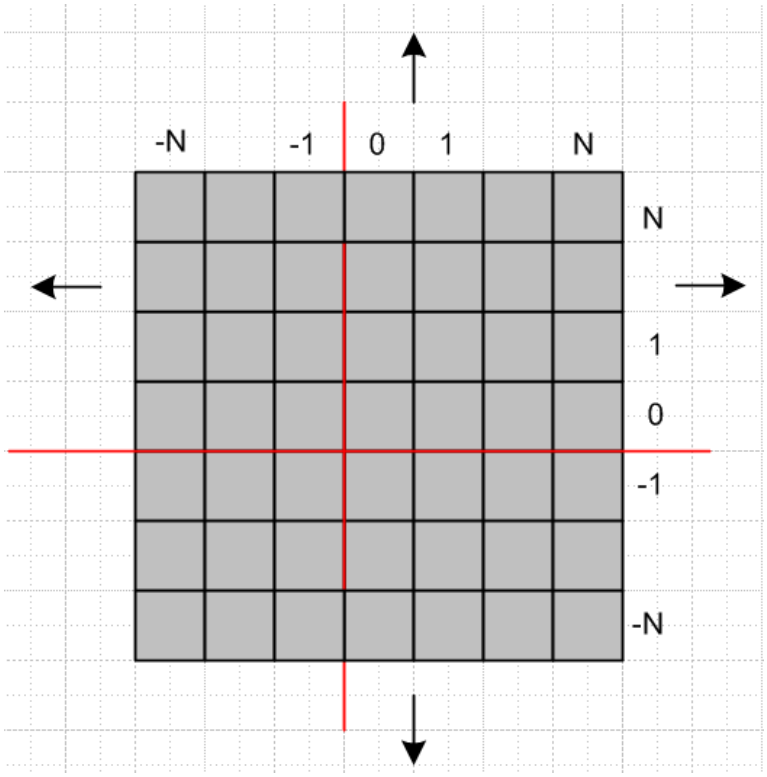
Then the double sum above becomes,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega(m-n)T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{JP+m'})^* \, (y_{IP+n'}) \, e^{i\omega P(I-J)T_1} \, e^{i\omega(m'-n')T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{m'})^* (y_{n'})\, e^{i\omega P(I-J)T_1}\, e^{i\omega(m'-n')T_1} \quad , \qquad \text{(F.2)}$$

where in the last line we have used the periodicity of the y_n. The following illustration shows how, in this reorganization, we first sum over a square grid patch with n' and m', and then we sum over an array of those patches with I and J.



In order to regulate things, we shall assume that the I and J sums range from -N to N rather than from -∞ to ∞. This means we are assuming that the sequence is (2N+1) repeated periods in length and not infinite. Then of course we can say

$$T = (2N+1)PT_1 \, .$$

As usual, we keep N finite as long as possible, and then take $N \to \infty$ in the end.

We now rewrite (F.2) by removing the primes from summation indices and reordering the factors

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \;\rightarrow$$

$$= \Big[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \Big] \Big[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \Big] \quad . \qquad \text{(F.3)}$$

The expression is now completely factored into two separate factors (thanks to the periodicity of y_n).

The second factor we recognize as

$$\left[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \right] = \left| \sum_{n=0}^{P-1} y_n \, e^{-i\omega n T_1} \right|^2 = |\, Y''(z) \,|^2 \qquad \text{(F.4)}$$

where $z = e^{ik\omega T_1}$ and X"(z) is the Z transform of our subsequence of length P.

The first factor can be processed using various results of Appendix A. We write

$$\left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] = \left| \sum_{I=-N}^{N} e^{i\omega P I T_1} \right|^2 \quad . \qquad \text{(F.5)}$$

Then apply (A.30)

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \, \delta_5(k,N) = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \qquad \textit{(A.30)}$$

with $k = \omega P T_1$ to get

$$\left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] = \left| \, 2\pi \, \delta_5(\omega P T_1, N) \, \right|^2 = \left[\, 2\pi \, \delta_5(\omega P T_1, N) \, \right]^2 . \qquad \text{(F.6)}$$

Then from (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \, \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1} \, \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \quad . \qquad \textit{(A.20)}$$

we can write the first factor of (F.3) as

$$\left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] = (2N+1)\, 2\pi \, \delta_6(\omega P T_1, N) \, . \qquad \text{(F.7)}$$

At this point we have

$$\mathcal{P}(\omega) = T_1 \, \mathcal{P}_{pulse}(\omega) \frac{1}{T} \left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] \left[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \right]$$

$$= T_1 \, \mathcal{P}_{pulse}(\omega) \frac{1}{(2N+1)PT_1} \left[(2N+1)\, 2\pi \, \delta_6(\omega P T_1, N) \right] \left[\, |\, Y''(z) \,|^2 \, \right]$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{P}\;\;[2\pi\,\delta_{\mathbf{6}}(\omega PT_1,N)\,]\;\;[\;|\,Y''(z)\,|^{\mathbf{2}}\;]\quad . \tag{F.8}$$

Now finally we take $N\to\infty$ using (A.21)

$$\lim_{\mathbf{N}\to\infty}\delta_{\mathbf{6}}(k,N) \;=\; \sum_{m=-\infty}^{\infty}\delta(k-2\pi m) \tag{A.21}$$

with this result

$$\mathcal{P}(\omega) \;= \mathcal{P}_{\mathbf{pulse}}(\omega)\;\frac{1}{P}\;\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega PT_1-2\pi m)\;\;|\,Y''(z)\,|^{\mathbf{2}}\;. \tag{F.9}$$

Recalling that $\omega_1 = 2\pi/T_1$, we write

$$\delta(\omega PT_1-2\pi m) \;= (PT_1)^{-1}\,\delta(\omega - 2\pi m/(PT_1)) \;= (PT_1)^{-1}\,\delta(\omega - \omega_1 m/P)\quad , \tag{F.10}$$

so our expression for $\mathcal{P}(\omega)$ becomes

$$\mathcal{P}(\omega) \;= \mathcal{P}_{\mathbf{pulse}}(\omega)\;\frac{1}{P^2}\,\frac{2\pi}{T_1}\;\sum_{m=-\infty}^{\infty}\delta(\omega - \omega_1 m/P)\;\;|\,Y''(z)\,|^{\mathbf{2}}$$

or

$$\mathcal{P}(\omega) \;= \mathcal{P}_{\mathbf{pulse}}(\omega)\;\frac{1}{P^2}\;\omega_1\;\sum_{m=-\infty}^{\infty}\;|\,Y''(z)\,|^{\mathbf{2}}\;\delta(\omega - \omega_1 m/P)$$

$$Y''(z) \;=\; \sum_{n=0}^{P-1} y_{\mathbf{n}}\,e^{-i\omega n T_1}$$

$$|\,Y''(z)\,|^{\mathbf{2}} = [\;\sum_{n=0}^{P-1}\;\sum_{m=0}^{P-1}\;y_{\mathbf{m}}{}^{*}\,y_{\mathbf{n}}\;e^{i\omega(m-n)T_1}\;] \quad . \tag{F.11}$$

Using a normalized Z transform $Q''(z) \equiv Y''(z)/P$ we can write this as

$$\mathcal{P}(\omega) \;= \mathcal{P}_{\mathbf{pulse}}(\omega)\;\omega_1\;\sum_{m=-\infty}^{\infty}\;|\,Q''(z)\,|^{\mathbf{2}}\;\delta(\omega - \omega_1 m/P)$$

$$Q''(z) \;=\; \frac{1}{P}\;\sum_{n=0}^{P-1} y_{\mathbf{n}}\,e^{-i\omega n T_1}$$

$$|\,Q''(z)\,|^{\mathbf{2}} = \frac{1}{P^2}\;\sum_{n=0}^{P-1}\;\sum_{m=0}^{P-1}\;y_{\mathbf{m}}{}^{*}\,y_{\mathbf{n}}\;e^{i\omega(m-n)T_1} \quad . \tag{F.12}$$

The result is seen to be eminently finite and of course is entirely discrete, as befits the spectrum of any periodic function.

(b) Calculation for an Ensemble of such Pulse Trains subject to Certain Conditions.

Suppose now instead of a particular subsequence of length P, we have an ensemble of such subsequences (all of length P), which, when installed, yield an ensemble of infinite-length pulse trains. In each pulse train of this ensemble, there is a sequence of length P that is repeated an infinite number of times.

Then we apply the ensemble average <..> to (F.12) to get

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P^2} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m^*\, y_n \rangle\ e^{i\omega(m-n)T_1} \qquad \text{(F.13)}$$

At this point, *suppose it happens* that

$$\langle y_m^*\, y_n \rangle = \alpha \qquad \text{for } m \neq n$$

$$\langle y_m^*\, y_n \rangle = \beta \qquad \text{for } m = n \qquad \text{(F.14)}$$

where α and β are *independent* of the indices shown. Notice in the (F.13) sum that max(n-m) = P-1, so we don't have to worry about these indices differing by an integral multiple of P. We have now restricted our interest now to the sequence $a_0, a_1, \ldots a_{P-1}$.

Comment: Equ. (F.14) would be true if the random variables Y_m and Y_n were uncorrelated, but saying that α and β are independent of the indices covers a larger class of possibilities, including some in which α and β are in fact correlated. In other words, saying that α and β as shown above don't depend on the indices m and n does not imply that $\langle y_m^*\, y_n \rangle = \langle y_m^* \rangle \langle y_n \rangle$ which is the definition of being uncorrelated.

Then we can write

$$\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m^*\, y_n \rangle\ e^{i\omega(m-n)T_1} = \alpha \sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} + \beta \sum_{n=0}^{P-1} 1$$

$$= \alpha \sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} + \beta P\ . \qquad \text{(F.15)}$$

To evaluate the double sum, we write it as

$$\sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} = \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} e^{i\omega(m-n)T_1} - \sum_{n=0}^{P-1} \sum_{m=n} e^{i\omega(m-n)T_1}$$

$$= \Big| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \Big|^2 - \sum_{n=0}^{P-1} 1 = \Big| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \Big|^2 - P\ . \qquad \text{(F.16)}$$

Now let $\omega T_1 = k$ and consider

$$\sum_{n=0}^{P-1} e^{-ink} = \sum_{n=0}^{P-1} (e^{-ik})^n = \sum_{n=0}^{P-1} x^n = 1 + x + \ldots + x^{P-1} = \frac{1-x^P}{1-x}$$

$$= \frac{1-e^{-ikP}}{1-e^{-ik}} \quad . \tag{F.17}$$

With a Maple assist,

```
assume(k,real): assume(P,real);
f := (1-exp(-I*k*P))/(1-exp(-I*k));
```

$$f := \frac{1 - e^{(-I k P)}}{1 - e^{(-I k)}}$$

```
f1 := abs(f^2);
```

$$f1 := \frac{(1 - \cos(k P))^2 + \sin(k P)^2}{(1 - \cos(k))^2 + \sin(k)^2}$$

```
f2 := simplify(f1);
```

$$f2 := \frac{-1 + \cos(k P)}{-1 + \cos(k)}$$

```
f3 := subs(cos(k) = 1 - 2*sin(k/2)^2,f2);
```

$$f3 := -\frac{1}{2}\frac{-1 + \cos(k P)}{\sin\left(\frac{1}{2}k\right)^2}$$

```
f4 := subs(cos(k*P) = 1 - 2*sin(k*P/2)^2,f3);
```

$$f4 := \frac{\sin\left(\frac{1}{2}k P\right)^2}{\sin\left(\frac{1}{2}k\right)^2}$$

we see that

$$\left| \sum_{n=0}^{P-1} e^{-ink} \right|^2 = \left| \frac{1-e^{-ikP}}{1-e^{-ik}} \right|^2 = \frac{\sin^2(kP/2)}{\sin^2(k/2)} \quad . \tag{F.18}$$

But from (A.20)

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \quad . \tag{A.20}$$

setting P = 2N+1 we recognize $\frac{\sin^2(kP/2)}{\sin^2(k/2)}$ as one of our multiple delta function models, so then

$$\left| \sum_{n=0}^{P-1} e^{-ink} \right|^2 = \left| \frac{1-e^{-ikP}}{1-e^{-ik}} \right|^2 = \frac{\sin^2(kP/2)}{\sin^2(k/2)} = 2\pi\, P\, \delta_6\left(k, \frac{P-1}{2}\right) \quad . \tag{F.19}$$

Setting $k = \omega T_1$ and applying (F.19) to (F.16) we get

$$\sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} = \left| \sum_{n=0}^{P-1} e^{-i\omega n T_1} \right|^2 - P = 2\pi P \, \delta_6(\omega T_1, \frac{P-1}{2}) - P$$

$$= P \, [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1]. \qquad \text{(F.20)}$$

We now go back to (F.13)

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \qquad \textit{(F.13)}$$

and install (F.15) to get

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \, [\alpha \; \{ \sum_{n=0}^{P-1} \sum_{m \neq n} e^{i\omega(m-n)T_1} \} + \beta P \,] \, . \qquad \text{(F.21)}$$

Then we use (F.20) for {...} to get

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P^2} \, [\alpha \; \{ \; P \, [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1] \} + \beta P \,]$$

$$= \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P} \, [\alpha \; \{ \; [2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) - 1] \} + \beta \,]$$

or

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; \frac{1}{P} \, [(\beta - \alpha) \; + \alpha \, 2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) \,] \qquad \text{(F.22a)}$$

where

$$\delta_6(k,N) \equiv \frac{1}{2N+1} \, \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \qquad \Rightarrow \qquad 2\pi \, \delta_6(\omega T_1, \frac{P-1}{2}) = \frac{1}{P} \; \frac{\sin^2(P\omega T_1/2)}{\sin^2(\omega T_1/2)} \, .$$

Evaluating at the delta function hit values $\omega = \omega_1 m/P$ we find

$$2\pi \, \delta_6 \; = \frac{1}{P} \frac{\sin^2(m\pi)}{\sin^2(m\pi/P)} \; = \begin{cases} 0 & m \neq NP \\ P & m = NP \end{cases} \quad \text{for N = any integer} \; .$$

To get alternate forms for (F.22a), we first express each term as a separate sum,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; \omega_1 \, \frac{1}{P} \, [(\beta - \alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \; + \; \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \, 2\pi \delta_6(\omega T_1, \frac{P-1}{2}) \,]$$

Since only those m which are multiples of P contribute to the second sum in (F.22a), we may rewrite that second sum as follows,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \frac{1}{P} \left[(\beta-\alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) + \alpha \sum_{N=-\infty}^{\infty} \delta(\omega - \omega_1 N)\, P \right] \qquad // \ m = NP$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m) \right] \qquad \text{(F.22b)}$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \left[(\beta-\alpha) \frac{1}{P} \delta(\omega - \omega_1 m/P) + \alpha\, \delta(\omega - \omega_1 m) \right] \qquad \text{(F.22c)}$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \{ (\beta-\alpha)/P + \alpha \}\, \delta(\omega) \right]$$

(F.22d)

If $x_{\mathtt{pulse}}(t)$ is real, then by (7.4) $\mathcal{P}_{\mathtt{pulse}}(\omega)$ is an even function of ω, and we can then reflect the negative part of the sum to the positive side to get,

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{2}{P} \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m/P) + 2\alpha \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m) + \{ (\beta-\alpha)/P + \alpha \}\, \delta(\omega) \right]$$

(F.22e)

In all forms of (F.22) we have: $\alpha = \langle y_m^* y_n \rangle$ for $m \neq n$ $\beta = \langle y_n^* y_n \rangle$.

These slightly different forms of $\langle \mathcal{P}(\omega) \rangle$ are useful for different purposes. All results are valid for any finite integer P. Since each sequence in the ensemble is periodic with the same period P, the ensemble average spectrum is entirely discrete. The m≠0 sums include positive and negative integers.

Here is a graphical representation of (F.22d) drawn for some arbitrary $\mathcal{P}_{\mathtt{pulse}}(\omega)$ and P = 4. Each

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \{ (\beta-\alpha)/P + \alpha \}\, \delta(\omega) \right]$$

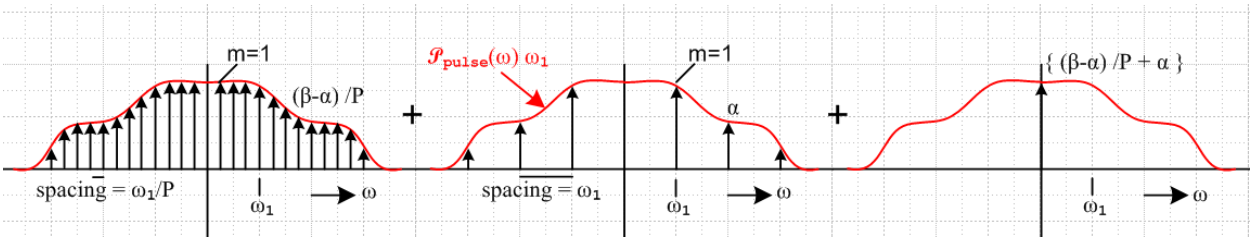
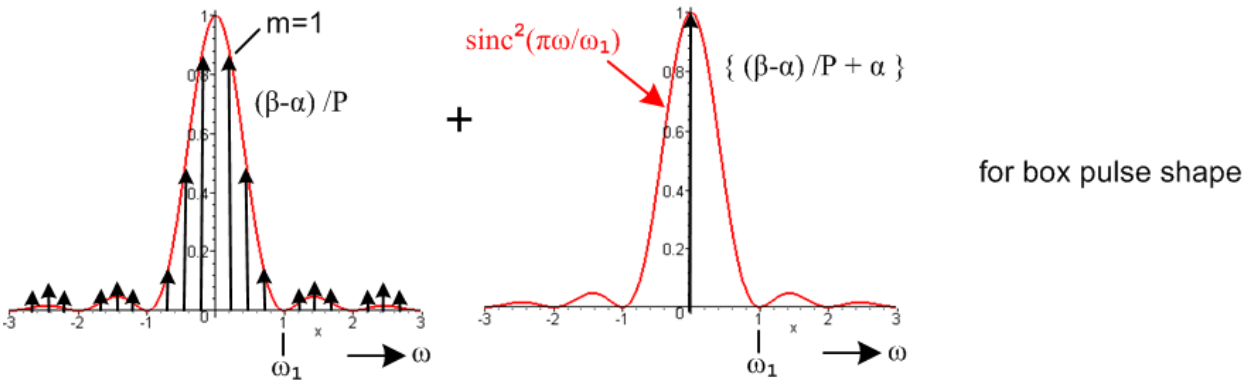


Fig F.1

Each vertical arrow represents a spectral δ line. The height of the arrow is the value of the red envelope curve times the quantity shown. The red curve $\mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1$ will in general have an infinite extent, an example being $\mathcal{P}_{\mathtt{pulse}}(\omega)\ \omega_1 = \mathrm{sinc}^2(\omega T_1/2) = \mathrm{sinc}^2(\pi\omega/\omega_1)$ for a box shaped pulse as in (9.2). Once a particular $\mathcal{P}_{\mathtt{pulse}}(\omega)$ is specified, it might quench some of the spectral lines. For the box example, lines are quenched when $\omega = N\omega_1$ for N = ±1,±2 .... In this case, all the lines of the central image go away and the corresponding lines in the left image also vanish, this being every $P^{\mathtt{th}}$ line in that image:



(c) Limit as P → ∞ of the Ensemble Result

We would now like to take the limit of the above as P→∞. To do this right, we would really have to undo the N→∞ limit, we can compute this limit as follows. Write (F.22b) as

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [(\beta-\alpha)\ \{ \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \}\ +\ \omega_1\ \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.23)}$$

Then define

$$f_P(\omega) \equiv \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ . \qquad \text{(F.24)}$$

As P→∞, the spacing of the δ lines becomes closer and closer, while the amplitude $\frac{\omega_1}{P}$ of each δ line becomes less and less. Perhaps we can argue that in the limit this becomes some continuous function.

In line with the distribution theory approach to symbolic functions noted in Appendix A, suppose we integrate this function from some a to a+ε for small ε, where a is an arbitrary real number,

$$\int_a^{a+\varepsilon} f_P(\omega)\, d\omega = \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \int_a^{a+\varepsilon} \delta(\omega - \omega_1 m/P)$$

$$= \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \Theta(a \le \omega_1 m/P \le a+\varepsilon) \qquad \text{(F.25)}$$

where we use the notation of Appendix A (e) for the Θ function which takes value 1 if the inequality argument is valid. If P is a large integer, how many non-zero terms does this Σ_m have? The inequality argument reads

$$Pa/\omega_1 \le m \le Pa/\omega_1 + P\varepsilon/\omega_1\ .$$

Since P is large, we round each term in this equation to the nearest integer, making little error. We select a very small ε first, and then we make sure P is large enough so $P\varepsilon/\omega_1$ is still a reasonably large integer when rounded. Then we have

$$\int_{a}^{a+\varepsilon} f_P(\omega)\, d\omega = \frac{\omega_1}{P} \sum_{m=Pa/\omega_1}^{Pa/\omega_1 + P\varepsilon/\omega_1} \Theta(a \le \omega_1 m/P \le a+\varepsilon) = \frac{\omega_1}{P} \sum_{m=Pa/\omega_1}^{Pa/\omega_1 + P\varepsilon/\omega_1} 1$$

$$= \frac{\omega_1}{P} \; (P\varepsilon/\omega_1) = \varepsilon \, . \qquad \text{(F.26)}$$

Since we then have (for very large P) that $\int_{a}^{a+\varepsilon} f_P(\omega)\, d\omega = \varepsilon$ for any real a and for ε as small as we like, and since the integral over range ε is proportional to ε, the function $f_P(\omega)$ is equivalent to the constant function 1. Thus we have shown that,

$$\lim_{P\to\infty} f_P(\omega) = \lim_{P\to\infty} [\, \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)] = 1. \qquad \text{(F.27)}$$

We then obtain this P→∞ limit of (F.23),

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; [(\beta-\alpha) + \omega_1 \; \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m) \;] \qquad \text{(F.28)}$$

$$= \mathcal{P}_{pulse}(\omega) \; [(\beta-\alpha) + (1/T_1) \; \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_1 m) \;] \qquad // \; \omega_1 = 2\pi/T_1$$

$$= \mathcal{P}_{pulse}(\omega) \; [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \;] \, .$$

This agrees with our result obtained earlier for a random ensemble of infinite sequences for which $\langle a_m a_n \rangle$ does not depend on the values of m and n,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \; \{ \, (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \, \} \qquad \textit{(35.11)}$$