

5.5 The Expected Value of a Function of Random Variables

The material in this section is combined with the material in the next section and given in the next section.

5.6 Special Theorems

- Expected value of a function of random variables, $E[g(Y_1, Y_2, \dots, Y_k)]$, equals

$$= \begin{cases} \sum_{\text{all } y_1} \sum_{\text{all } y_2} \cdots \sum_{\text{all } y_k} g(y_1, y_2, \dots, y_k) p(y_1, y_2, \dots, y_k) & \text{if discrete,} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(y_1, y_2, \dots, y_k) f(y_1, y_2, \dots, y_k) dy_1 dy_2 \cdots dy_k & \text{if continuous.} \end{cases}$$

- Also, the *variance* of a function of random variables, $V[g(Y_1, Y_2, \dots, Y_k)]$, with expected value, $E[g(Y_1, Y_2, \dots, Y_k)]$, is defined

$$\begin{aligned} V[g(Y_1, Y_2, \dots, Y_k)] &= E[(g(Y_1, Y_2, \dots, Y_k) - E[g(Y_1, Y_2, \dots, Y_k)])^2] \\ &= E[g(Y_1, Y_2, \dots, Y_k)^2] - [E[g(Y_1, Y_2, \dots, Y_k)]]^2. \end{aligned}$$

- Related to this,

$$\begin{aligned} E[Y_1 + Y_2 + \cdots + Y_k] &= E[Y_1] + \cdots + E[Y_k], \\ E[g_1(Y_1, Y_2) + \cdots + g_k(Y_1, Y_2)] &= E[g_1(Y_1, Y_2)] + \cdots + E[g_k(Y_1, Y_2)], \\ E(c) &= c, \quad \text{if } c \text{ is a constant,} \\ E[cg(Y_1, Y_2)] &= cE[g(Y_1, Y_2)], \quad \text{if } c \text{ is a constant,} \\ E[g(Y_1)h(Y_2)] &= E[g(Y_1)]E[h(Y_2)], \quad \text{if } g(Y_1), h(Y_2) \text{ independent.} \end{aligned}$$

Exercise 5.6 (Special Theorems)

- Discrete Expected Value Calculations: Waiting Times To Catch Fish.* The joint density, $p(y_1, y_2)$, of the number of minutes waiting to catch the *first* fish, y_1 , and the number of minutes waiting to catch the *second* fish, y_2 , is given below.

$y_2 \downarrow y_1 \rightarrow$	1	2	3	total
1	0.01	0.01	0.07	0.09
2	0.02	0.02	0.08	0.12
3	0.08	0.08	0.63	0.79
total	0.11	0.11	0.78	1.00

- (a) The *expected* average waiting time, $g(y_1, y_2) = \frac{y_1 + y_2}{2}$, over two trips is

$$\begin{aligned}
 E \left[\frac{Y_1 + Y_2}{2} \right] &= \sum_{y_1=1}^3 \sum_{y_2=1}^3 \left(\frac{y_1 + y_2}{2} \right) p(y_1, y_2) \\
 &= \left(\frac{1+1}{2} \right) (0.01) + \left(\frac{1+2}{2} \right) (0.02) + \left(\frac{1+3}{2} \right) (0.08) \\
 &\quad + \left(\frac{2+1}{2} \right) (0.01) + \left(\frac{2+2}{2} \right) (0.02) + \left(\frac{2+3}{2} \right) (0.08) \\
 &\quad + \left(\frac{3+1}{2} \right) (0.07) + \left(\frac{3+2}{2} \right) (0.08) + \left(\frac{3+3}{2} \right) (0.63) =
 \end{aligned}$$

(choose one) (i) **2.385** (ii) **2.685** (iii) **2.785**.

(Hint: Type 1, 1, 1, 2, 2, 2, 3, 3, 3 in L_1 and 1, 2, 3, 1, 2, 3, 1, 2, 3 in L_2 , define $L_3 = \frac{L_1 + L_2}{2}$, type 0.01, 0.02, 0.08, 0.01, 0.02, 0.08, 0.07, 0.07, 0.63 in L_4 , define $L_5 = L_3 \times L_4$, sum L_5 using 1-Var Stats L_5 , read $\sum x = 2.685$.)

- (b) The expected *total* waiting time, $g(y_1, y_2) = y_1 + y_2$, over two trips is

$$\begin{aligned}
 E[Y_1 + Y_2] &= \sum_{y_1=1}^3 \sum_{y_2=1}^3 (y_1 + y_2) p(y_1, y_2) \\
 &= (1 + 1) (0.01) + (1 + 2) (0.02) + (1 + 3) (0.08) \\
 &\quad + (2 + 1) (0.01) + (2 + 2) (0.02) + (2 + 3) (0.08) \\
 &\quad + (3 + 1) (0.07) + (3 + 2) (0.08) + (3 + 3) (0.63) =
 \end{aligned}$$

(choose one) (i) **5.37** (ii) **6.37** (iii) **7.37**.

(Hint: Define $L_3 = L_1 + L_2$, define $L_5 = L_3 \times L_4$, sum L_5 using 1-Var Stats L_5 , read $\sum x = 5.37$.)

- (c) Marginal probability function for Y_1 is given by column totals; in this case:

y_1	1	2	3
$p_1(y_1)$	0.11	0.11	0.78

so expected waiting time for *first* trip is

$$E[Y_1] = \sum_{y_1=1}^3 y_1 p_1(y_1) = (1)(0.11) + (2)(0.11) + (3)(0.78) =$$

(choose one) (i) **1.67** (ii) **2.67** (iii) **3.67**.

- (d) Marginal probability function for Y_2 is given by row totals; in this case:

y_2	1	2	3
$p_2(y_2)$	0.09	0.12	0.79

so expected waiting time for *second* trip is

$$E[Y_2] = \sum_{y_2=1}^3 y_2 p_2(y_2) = (1)(0.09) + (2)(0.12) + (3)(0.79) =$$

(choose one) (i) **1.7** (ii) **2.7** (iii) **3.7**.

(e) Notice

$$E[Y_1 + Y_2] = 5.37 = E[Y_1] + E[Y_2] = 2.67 + 2.7$$

(choose one) (i) **True** (ii) **False**

(f) Since

$$E[Y_2^2] = \sum_{y_2=1}^3 y_2^2 p_2(y_2) = (1^2)(0.09) + (2^2)(0.12) + (3^2)(0.79) = 7.68,$$

then *variance* in *second* waiting time is

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = 7.68 - 2.7^2 =$$

(choose one) (i) **0.29** (ii) **0.39** (iii) **0.49**.

2. *Discrete Expectation Value Calculations: Marbles In An Urn.* Marbles chosen at random *without* replacement from an urn consisting of 8 blue and 6 black marbles. For i th marble chosen, $Y_i = 0$ if blue and $Y_i = 1$ if black.

$y_2 \downarrow y_1 \rightarrow$	blue, 0	black, 1	$p_2(y_2)$
blue, 0	$\frac{8 \cdot 7}{14 \cdot 13}$	$\frac{6 \cdot 8}{14 \cdot 13}$	$\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13}$
black, 1	$\frac{8 \cdot 6}{14 \cdot 13}$	$\frac{6 \cdot 5}{14 \cdot 13}$	$\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13}$
$p_1(y_1)$	$\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13}$	$\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13}$	1

(a) The expected value of $g(y_1, y_2) = y_1 y_2$ is

$$\begin{aligned} E[Y_1 Y_2] &= \sum_{y_1=0}^1 \sum_{y_2=0}^1 (y_1 y_2) p(y_1, y_2) \\ &= (0 \times 0) \left(\frac{8 \cdot 7}{14 \cdot 13} \right) + (0 \times 1) \left(\frac{8 \cdot 6}{14 \cdot 13} \right) \\ &\quad + (1 \times 0) \left(\frac{8 \cdot 8}{14 \cdot 13} \right) + (1 \times 1) \left(\frac{6 \cdot 5}{14 \cdot 13} \right) = \end{aligned}$$

(choose one) (i) $\frac{8 \cdot 6}{14 \cdot 13}$ (ii) $\frac{8 \cdot 7}{14 \cdot 13}$ (iii) $\frac{6 \cdot 5}{14 \cdot 13}$.

(b) Expected value of Y_1 is

$$E[Y_1] = \sum_{y_1=0}^1 y_1 p_1(y_1) = (0) \left(\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13} \right) + (1) \left(\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} \right) =$$

(choose one) (i) $\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13}$ (ii) $\frac{8 \cdot 7 + 2 \times 8 \cdot 6}{14 \cdot 13}$ (iii) $\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13}$.

(c) Expected value of Y_2 is

$$E[Y_2] = \sum_{y_2=0}^1 y_2 p_2(y_2) = (0) \left(\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13} \right) + (1) \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) =$$

(choose one) (i) $\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13}$ (ii) $\frac{8 \cdot 7 + 2 \times 8 \cdot 6}{14 \cdot 13}$ (iii) $\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13}$.

(Notice, in this case, $E[Y_1] = E[Y_2]$.)

(d) In this case,

$$E[Y_1 Y_2] = \frac{6 \cdot 5}{14 \cdot 13} \neq E[Y_1] E[Y_2] = \left(\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} \right) \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)$$

because Y_1 and Y_2 are *dependent*.

(choose one) (i) **True** (ii) **False**

(e) Using the properties of expectation above,

$$E[3Y_1 - 2Y_2] = 3E[Y_1] - 2E[Y_2] = 3 \left(\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} \right) - 2 \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) =$$

(i) $\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13}$ (ii) $\frac{8 \cdot 7 + 2 \times 8 \cdot 6}{14 \cdot 13}$ (iii) $\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13}$.

(f) Since

$$E[Y_2^2] = \sum_{y_2=0}^0 y_2^2 p_2(y_2) = (0^2) \left(\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13} \right) + (1^2) \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) = \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13},$$

then the variance is

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} - \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)^2 \approx$$

(choose one) (i) **0.245** (ii) **0.345** (iii) **0.445**.

(g) The definition of variance is

$$\begin{aligned} V[Y_2] &= E[(Y_2 - E[Y_2])^2] \\ &= \sum_{y_2=0}^1 (Y_2 - E[Y_2])^2 p_2(y_2) \\ &= \left(0 - \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)^2 \left(\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13} \right) + \left(1 - \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)^2 \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) \approx \end{aligned}$$

(choose one) (i) **0.245** (ii) **0.345** (iii) **0.445**.

3. *Continuous Expectation Value Calculations: Potato Chips.* Although each bag should weigh 50 grams each and contain 5 milligrams of salt, in fact, because of differing machines, weight and amount of salt placed in each bag varies according to two probability functions below.

(a) *Machine A.* Bivariate density function for machine A is

$$f(y_1, y_2) = \begin{cases} \frac{1}{12}, & 49 \leq y_1 \leq 51, 2 \leq y_2 \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

- i. The expected value of $g(y_1, y_2) = y_1 y_2$ is

$$\begin{aligned} E[Y_1 Y_2] &= \int_2^8 \int_{49}^{51} (y_1 y_2) f(y_1, y_2) dy_1 dy_2 \\ &= \int_2^8 \int_{49}^{51} (y_1 y_2) \frac{1}{12} dy_1 dy_2 \\ &= \int_2^8 \frac{1}{12} y_2 \left(\frac{1}{2} y_1^2 \right)_{y_1=49}^{y_1=51} dy_2 \\ &= \left(\frac{100}{24} y_2^2 \right)_{y_1=2}^{y_1=8} = \end{aligned}$$

(choose one) (i) **200** (ii) **250** (iii) **300**.

- ii. Since marginal,

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_2^8 \frac{1}{12} dy_2 = \frac{1}{12} (y_2)_{y_2=2}^{y_2=8} = \frac{1}{2},$$

$49 \leq y_1 \leq 51$, expected value of Y_1 is

$$E[Y_1] = \int_{49}^{51} y_1 f_1(y_1) dy_1 = \int_{49}^{51} y_1 \frac{1}{2} dy_1 = \left(\frac{1}{4} y_1^2 \right)_{y_1=49}^{y_1=51} =$$

(choose one) (i) **50** (ii) **100** (iii) **150**.

- iii. Since marginal,

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 = \int_{49}^{51} \frac{1}{12} dy_1 = \frac{1}{12} (y_1)_{y_1=49}^{y_1=51} = \frac{1}{6},$$

$2 \leq y_2 \leq 8$, expected value of Y_2 is

$$E[Y_2] = \int_2^8 y_2 f_2(y_2) dy_2 = \int_2^8 y_2 \frac{1}{6} dy_2 = \left(\frac{1}{12} y_2^2 \right)_{y_2=2}^{y_2=8} =$$

(choose one) (i) **5** (ii) **10** (iii) **15**.

iv. Since

$$E[Y_1 Y_2] = 250 = E[Y_1]E[Y_2] = (50)(5)$$

Y_1 and Y_2 are (choose one) (i) **dependent** (ii) **independent**.

v. Using the properties of expectation above,

$$E[3Y_1 - 2Y_2] = 3E[Y_1] - 2E[Y_2] = 3(50) - 2(5) =$$

(i) **130** (ii) **140** (iii) **150**.

vi. Since

$$E[Y_2^2] = \int_2^8 y_2^2 f_2(y_2) dy_2 = \int_2^8 y_2^2 \frac{1}{6} dy_2 = \left(\frac{1}{18} y_2^3 \right)_{y_2=2}^{y_2=8} = 28,$$

then variance is

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = 28 - (5)^2 \approx$$

(choose one) (i) **1** (ii) **2** (iii) **3**.

(b) *Machine B*. Bivariate density function for machine B is

$$f(y_1, y_2) = \begin{cases} \frac{1}{4}, & 49 \leq y_1 \leq 51, 4 \leq y_2 \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

i. The expected value of $g(y_1, y_2) = y_1 y_2$ is

$$\begin{aligned} E[Y_1 Y_2] &= \int_4^6 \int_{49}^{51} (y_1 y_2) f(y_1, y_2) dy_1 dy_2 \\ &= \int_4^6 \int_{49}^{51} (y_1 y_2) \frac{1}{4} dy_1 dy_2 \\ &= \int_4^6 \frac{1}{4} y_2 \left(\frac{1}{2} y_1^2 \right)_{y_1=49}^{y_1=51} dy_2 \\ &= \left(\frac{100}{8} y_2^2 \right)_{y_2=4}^{y_2=6} = \end{aligned}$$

(choose one) (i) **250** (ii) **500** (iii) **750**.

ii. Since marginal,

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 = \int_4^6 \frac{1}{4} dy_2 = \frac{1}{4} (y_2)_{y_2=4}^{y_2=6} = \frac{1}{2},$$

$49 \leq y_1 \leq 51$, expected value of Y_1 is

$$E[Y_1] = \int_{49}^{51} y_1 f_1(y_1) dy_1 = \int_{49}^{51} y_1 \frac{1}{2} dy_1 = \left(\frac{1}{4} y_1^2 \right)_{y_1=49}^{y_1=51} =$$

(choose one) (i) **50** (ii) **100** (iii) **150**.

iii. Since marginal,

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 = \int_{49}^{51} \frac{1}{4} dy_1 = \frac{1}{4}(y_1)_{y_1=49}^{y_1=51} = \frac{1}{2},$$

$4 \leq y_2 \leq 6$, expected value of Y_2 is

$$E[Y_2] = \int_4^6 y_2 f_2(y_2) dy_2 = \int_4^6 y_2 \frac{1}{2} dy_2 = \left(\frac{1}{4} y_2^2 \right)_{y_2=4}^{y_2=6} =$$

(choose one) (i) **5** (ii) **10** (iii) **15**.

iv. Since

$$E[Y_1 Y_2] = 250 = E[Y_1] E[Y_2] = (50)(5)$$

Y_1 and Y_2 are (choose one) (i) **dependent** (ii) **independent**.

v. Using the properties of expectation above,

$$E[3Y_1 + 3Y_2] = 3E[Y_1] + 3E[Y_2] = 3(50) + 3(5) =$$

(i) **155** (ii) **160** (iii) **165**.

vi. Since

$$E[Y_2^2] = \int_4^6 y_2^2 f_2(y_2) dy_2 = \int_4^6 y_2^2 \frac{1}{2} dy_2 = \left(\frac{1}{6} y_2^3 \right)_{y_2=4}^{y_2=6} = \frac{76}{3},$$

then variance is

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = \frac{76}{3} - (5)^2 \approx$$

(choose one) (i) $\frac{1}{3}$ (ii) $\frac{2}{3}$ (iii) $\frac{3}{3}$.

vii. Let $U = Y_1 - Y_2$ and assume Y_1 and Y_2 are independent.

$$\begin{aligned} V(U) &= E(U^2) - E(U)^2 \\ &= E[(Y_1 - Y_2)^2] - [E(Y_1 - Y_2)]^2 \\ &= E[Y_1^2 - 2Y_1Y_2 + Y_2^2] - [E(Y_1) - E(Y_2)]^2 \\ &= E[Y_1^2] - 2E[Y_1Y_2] + E[Y_2^2] - \{[E(Y_1)]^2 - 2E[Y_1]E[Y_2] + [E(Y_2)]^2\} \\ &= E[Y_1^2] - 2E[Y_1Y_2] + E[Y_2^2] - \{[E(Y_1)]^2 - 2E[Y_1Y_2] + [E(Y_2)]^2\} \\ &= E[Y_1^2] - [E(Y_1)]^2 + \{E[Y_2^2] - [E(Y_2)]^2\} \\ &= V(Y_1) + V(Y_2) \end{aligned}$$

(i) **True** (ii) **False**

4. *More Continuous Expectation Value Calculations.*

Consider

$$f(y_1, y_2) = \begin{cases} \frac{6}{5}(2 - y_1)(1 - y_2) & 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1, y_1 + 2y_2 < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Since $y_1 + 2y_2 \leq 2$, then $y_2 \leq 1 - \frac{1}{2}y_1$, and marginal of Y_1 is

$$\begin{aligned} \frac{6}{5} \int_0^{1-\frac{1}{2}y_1} (2 - y_1)(1 - y_2) dy_2 &= \frac{6}{5} \int_0^{1-\frac{1}{2}y_1} (2 - y_1 - 2y_2 + y_1y_2) dy_2 \\ &= \frac{6}{5} \left(2y_2 - y_1y_2 - y_2^2 + \frac{1}{2}y_1y_2^2 \right)_{y_2=0}^{y_2=1-\frac{1}{2}y_1} \\ &= \frac{6}{5} \left(1 - \frac{1}{2}y_1 - \frac{1}{4}y_1^2 + \frac{1}{8}y_1^3 \right) \end{aligned}$$

where $0 \leq y_1 \leq 2$. So, the expected value of Y_1 is

$$\begin{aligned} E[Y_1] &= \int_0^2 y_1 f_1(y_1) dy_1 = \frac{6}{5} \int_0^2 y_1 \left(1 - \frac{1}{2}y_1 - \frac{1}{4}y_1^2 + \frac{1}{8}y_1^3 \right) dy_1 \\ &= \frac{6}{5} \left(\frac{1}{2}y_1^2 - \frac{1}{6}y_1^3 - \frac{1}{16}y_1^4 + \frac{1}{40}y_1^5 \right)_{y_1=0}^{y_1=2} = \end{aligned}$$

(choose one) (i) $\frac{13}{25}$ (ii) $\frac{14}{25}$ (iii) $\frac{15}{25}$.(b) Since $y_1 + 2y_2 \leq 2$, then $y_1 \leq 2 - 2y_2$, and marginal of Y_2 is

$$\begin{aligned} \frac{6}{5} \int_0^{2-2y_2} (2 - y_1)(1 - y_2) dy_1 &= \frac{6}{5} \int_0^{2-2y_2} (2 - y_1 - 2y_2 + y_1y_2) dy_1 \\ &= \frac{6}{5} \left(2y_1 - \frac{1}{2}y_1^2 - 2y_1y_2 + \frac{1}{2}y_1^2y_2 \right)_{y_1=0}^{y_1=2-2y_2} \\ &= \frac{6}{5} (2 - 2y_2 - 2y_2^2 + 2y_2^3) \end{aligned}$$

where $0 \leq y_2 \leq 1$. So, the expected value of Y_2 is

$$\begin{aligned} E[Y_2] &= \int_0^1 y_2 f_2(y_2) dy_2 = \frac{6}{5} \int_0^1 y_2 (2 - 2y_2 - 2y_2^2 + 2y_2^3) dy_2 \\ &= \frac{6}{5} \left(y_2^2 - \frac{2}{3}y_2^3 - \frac{2}{4}y_2^4 + \frac{2}{5}y_2^5 \right)_{y_2=0}^{y_2=1} = \end{aligned}$$

(choose one) (i) $\frac{7}{25}$ (ii) $\frac{8}{25}$ (iii) $\frac{9}{25}$.

(c) Using the properties of expectation above,

$$E[25Y_1 + 50Y_2] = 25E[Y_1] + 50E[Y_2] = 25 \left(\frac{17}{25} \right) + 50 \left(\frac{7}{25} \right) =$$

(i) **29** (ii) **30** (iii) **31**.

(d) Since

$$\begin{aligned} E[Y_2^2] &= \int_0^1 y_2^2 f_2(y_2) dy_2 = \frac{6}{5} \int_0^1 y_2^2 (2 - 2y_2 - 2y_2^2 + 2y_2^3) dy_2 \\ &= \frac{6}{5} \left(\frac{2}{3} y_2^3 - \frac{2}{4} y_2^4 - \frac{2}{5} y_2^5 + \frac{2}{6} y_2^6 \right)_{y_2=0}^{y_2=1} = \frac{3}{25}, \end{aligned}$$

then variance is

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = \frac{3}{25} - \left(\frac{7}{25} \right)^2 \approx$$

(choose one) (i) $\frac{24}{625}$ (ii) $\frac{25}{625}$ (iii) $\frac{26}{625}$.

(e) According to Tchebysheff's theorem, the probability variable Y_2 is *within* $k = 3$ standard deviations of mean $E[Y_2]$ is *at least*

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{3^2} = \frac{8}{9}.$$

That is, there is at least a $\frac{8}{9}$ th chance the following interval contains Y_2 :

$$E[Y_2] \pm 3\sqrt{V[Y_2]} = \frac{7}{25} \pm 3\sqrt{\frac{26}{625}} \approx$$

(i) **(-0.23, 0.79)** (ii) **(-0.33, 0.89)** (iii) **(-0.43, 0.99)**.

5. *Expected Number of Matches.* Ten people throw ten tickets with their names on each ticket into a jar, then draw one ticket out of the jar at random (and put it back in the jar). Let X be the number of people who select their own ticket out of the jar. Let

$$X = Y_1 + Y_2 + \cdots + Y_{10}$$

where

$$Y_i = \begin{cases} 1 & \text{if } i\text{th person selects own ticket} \\ 0 & \text{if } i\text{th person does not select their own ticket} \end{cases}$$

- (a) Each person chooses any of the ten tickets with equal chance,
 $E[Y_i] = 1 \times p(1) + 0 \times p(0) =$ (choose one) (i) $\frac{1}{10}$ (ii) $\frac{2}{10}$ (iii) $\frac{3}{10}$.
- (b) Expected number of ten individuals to choose their own ticket is
 $E(X) = E(Y_1) + \cdots + E(Y_{10}) = 10 \times \frac{1}{10} =$ (i) $\frac{8}{10}$ (ii) $\frac{9}{10}$ (iii) $\frac{10}{10}$.
 We would expect one of ten individuals to choose their own ticket.
- (c) If n individuals played this game, then we would expect
 $E(X) = E(Y_1) + \cdots + E(Y_n) = n \left(\frac{1}{n} \right) =$ (i) $\frac{n-1}{n}$ (ii) $\frac{n}{n}$ (iii) $\frac{n+1}{n}$.

5.7 The Covariance of Two Random Variables

We look at *covariance*, a measure of how two random variables are linearly related. Covariance is defined by

$$\text{Cov}(Y_1, Y_2) = E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))] = E(Y_1 Y_2) - E(Y_1)E(Y_2),$$

and has the following properties,

- $\text{Cov}(Y_1, Y_2) = \text{Cov}(Y_2, Y_1)$,
- $\text{Cov}(Y_1, Y_1) = V(Y_1)$,
- $\text{Cov}(aY_1, Y_2) = a\text{Cov}(Y_1, Y_2)$, where a is a constant,
- $\text{Cov}(Y_1, Y_2) = 0$ if Y_1, Y_2 are uncorrelated or independent.

Uncorrelated random variables are *not* necessarily independent random variables. Independent random variables *are* necessarily uncorrelated random variables. The *correlation* ρ , $-1 \leq \rho \leq 1$, is given by

$$\rho(y_1, y_2) = \rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{\text{Cov}(Y_1, Y_2)}{\sigma_1 \sigma_2}.$$

Exercise 5.7 (The Covariance of Two Random Variables)

1. *Covariance and Correlation: Waiting Times To Catch Fish.* The joint density, $p(y_1, y_2)$, of the number of minutes waiting to catch the *first* fish, y_1 , and the number of minutes waiting to catch the *second* fish, y_2 , is given below.

$y_2 \downarrow y_1 \rightarrow$	1	2	3	total
1	0.01	0.01	0.07	0.09
2	0.02	0.02	0.08	0.12
3	0.08	0.08	0.63	0.79
total	0.11	0.11	0.78	1.00

- (a) If $g(y_1, y_2) = y_1 y_2$, then

$$\begin{aligned}
 E[Y_1 Y_2] &= \sum_{y_1=1}^3 \sum_{y_2=1}^3 (y_1 y_2) p(y_1, y_2) \\
 &= (1 \times 1)(0.01) + (1 \times 2)(0.02) + (1 \times 3)(0.08) \\
 &\quad + (2 \times 1)(0.01) + (2 \times 2)(0.02) + (2 \times 3)(0.08) \\
 &\quad + (3 \times 1)(0.07) + (3 \times 2)(0.08) + (3 \times 3)(0.63) =
 \end{aligned}$$

(choose one) (i) **5.23** (ii) **6.23** (iii) **7.23**.

(Hint: Type 1, 1, 1, 2, 2, 2, 3, 3, 3 in L_1 and 1, 2, 3, 1, 2, 3, 1, 2, 3 in L_2 , define $L_3 = L_1 \times L_2$, type 0.01, 0.02, 0.08, 0.01, 0.02, 0.08, 0.07, 0.08, 0.63 in L_4 , define $L_5 = L_3 \times L_4$, sum L_5 using 1-Var Stats L_5 , read $\sum x = 7.23$.)

(b) Since

$$E[Y_1] = \sum_{y_1=1}^3 y_1 p_1(y_1) = (1)(0.11) + (2)(0.11) + (3)(0.78) = 2.67$$

$$E[Y_2] = \sum_{y_2=1}^3 y_2 p_2(y_2) = (1)(0.09) + (2)(0.12) + (3)(0.79) = 2.7,$$

then covariance is

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 7.23 - (2.67)(2.7) \approx$$

(choose one) (i) **-0.039** (ii) **0.021** (iii) **0.139**.

(c) Since

$$E[Y_1^2] = \sum_{y_1=1}^3 y_1^2 p_1(y_1) = (1^2)(0.11) + (2^2)(0.11) + (3^2)(0.78) = 7.57,$$

$$V[Y_1] = E[Y_1^2] - [E(Y_1)]^2 = 7.57 - 2.67^2 = 0.4411,$$

$$E[Y_2^2] = \sum_{y_2=1}^3 y_2^2 p_2(y_2) = (1^2)(0.09) + (2^2)(0.12) + (3^2)(0.79) = 7.68,$$

$$V[Y_2] = E[Y_2^2] - [E(Y_2)]^2 = 7.68 - 2.7^2 = 0.39,$$

then correlation is

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{-0.039}{\sqrt{0.4411 \times 0.39}} \approx$$

(choose one) (i) **-0.235** (ii) **-0.139** (iii) **-0.094**.

There is very little linear correlation between two waiting times.

(d) Let $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$. Then

$$\begin{aligned} \text{Cov}(U_1, U_2) &= E(U_1 U_2) - E(U_1)E(U_2) \\ &= E[(Y_1 + Y_2)(Y_1 - Y_2)] - E((Y_1 + Y_2))E((Y_1 - Y_2)) \\ &= E[Y_1^2 - Y_2^2] - [E(Y_1) + E(Y_2)][E(Y_1) - E(Y_2)] \\ &= E[Y_1^2] - E[Y_2^2] - \{[E(Y_1)]^2 - [E(Y_2)]^2\} \\ &= E[Y_1^2] - [E(Y_1)]^2 - \{E[Y_2^2] - [E(Y_2)]^2\} \\ &= V(Y_1) - V(Y_2) = 0.4411 - 0.39 \approx \end{aligned}$$

(choose one) (i) **0.0355** (ii) **0.0392** (iii) **0.0511**.

2. *Covariance and Correlation: Marbles In An Urn.* Marbles chosen at random without replacement from an urn consisting of 8 blue and 6 black marbles. For i th marble chosen, $Y_i = 0$ if blue and $Y_i = 1$ if black.

$y_2 \downarrow y_1 \rightarrow$	blue, 0	black, 1	$p_2(y_2)$
blue, 0	$\frac{8 \cdot 7}{14 \cdot 13}$	$\frac{6 \cdot 8}{14 \cdot 13}$	$\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13}$
black, 1	$\frac{8 \cdot 6}{14 \cdot 13}$	$\frac{6 \cdot 5}{14 \cdot 13}$	$\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13}$
$p_1(y_1)$	$\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13}$	$\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13}$	1

(a) Since

$$\begin{aligned}
 E[Y_1 Y_2] &= \sum_{y_1=0}^1 \sum_{y_2=0}^1 (y_1 y_2) p(y_1, y_2) \\
 &= (0 \times 0) \left(\frac{8 \cdot 7}{14 \cdot 13} \right) + (0 \times 1) \left(\frac{8 \cdot 6}{14 \cdot 13} \right) \\
 &\quad + (1 \times 0) \left(\frac{8 \cdot 8}{14 \cdot 13} \right) + (1 \times 1) \left(\frac{6 \cdot 5}{14 \cdot 13} \right) = \frac{6 \cdot 5}{14 \cdot 13}, \\
 E[Y_1] &= \sum_{y_1=0}^1 y_1 p_1(y_1) \\
 &= (0) \left(\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13} \right) + (1) \left(\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} \right) = \frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13}, \\
 E[Y_2] &= \sum_{y_2=0}^1 y_2 p_2(y_2) \\
 &= (0) \left(\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13} \right) + (1) \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) = \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13},
 \end{aligned}$$

then covariance is

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) =$$

(choose one) (i) $-\frac{12}{637}$ (ii) $-\frac{13}{637}$ (iii) $-\frac{14}{637}$.

(b) Since

$$E[Y_1^2] = \sum_{y_1=1}^3 y_1^2 p_1(y_1) = (0^2) \left(\frac{8 \cdot 7 + 8 \cdot 6}{14 \cdot 13} \right) + (1^2) \left(\frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} \right) = \frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13},$$

$$V[Y_1] = E[Y_1^2] - [E[Y_1]]^2 = \frac{6 \cdot 8 + 6 \cdot 5}{14 \cdot 13} - \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)^2 \approx 0.245,$$

$$E[Y_2^2] = \sum_{y_2=0}^0 y_2^2 p_2(y_2) = (0^2) \left(\frac{8 \cdot 7 + 6 \cdot 8}{14 \cdot 13} \right) + (1^2) \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right) = \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13},$$

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = \frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} - \left(\frac{8 \cdot 6 + 6 \cdot 5}{14 \cdot 13} \right)^2 \approx 0.245,$$

then correlation is

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{-\frac{12}{637}}{\sqrt{0.245 \times 0.245}} \approx$$

(choose one) (i) **-0.077** (ii) **-0.139** (iii) **-0.294**.

(c) *Relationship between covariance and variance.*

$$\text{Cov}(Y_1, Y_1) = E(Y_1 Y_1) - E(Y_1)E(Y_1) = E(Y_1^2) - [E(Y_1)]^2 = V(Y_1) \approx 0.245$$

$$\text{Cov}(Y_2, Y_2) = E(Y_2 Y_2) - E(Y_2)E(Y_2) = E(Y_2^2) - [E(Y_2)]^2 = V(Y_2) \approx 0.245$$

(i) **True** (ii) **False**

3. *Covariance and Correlation: Weight and Amount of Salt in Potato Chips.*

(a) *Machine A.* Bivariate density function for machine A is

$$f(y_1, y_2) = \begin{cases} \frac{1}{12}, & 49 \leq y_1 \leq 51, 2 \leq y_2 \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

i. Since

$$\begin{aligned}
 E[Y_1 Y_2] &= \int_2^8 \int_{49}^{51} (y_1 y_2) f(y_1, y_2) dy_1 dy_2 \\
 &= \int_2^8 \int_{49}^{51} (y_1 y_2) \frac{1}{12} dy_1 dy_2 = \int_2^8 \frac{1}{12} y_2 \left(\frac{1}{2} y_1^2 \right)_{y_1=49}^{y_1=51} dy_2 \\
 &= \left(\frac{100}{24} y_2^2 \right)_{y_2=2}^{y_2=8} = 250, \\
 f_1(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 \\
 &= \int_2^8 \frac{1}{12} dy_2 = \frac{1}{12} (y_2)_{y_2=2}^{y_2=8} = \frac{1}{2}, \\
 E[Y_1] &= \int_{49}^{51} y_1 f_1(y_1) dy_1 \\
 &= \int_{49}^{51} y_1 \frac{1}{2} dy_1 = \left(\frac{1}{4} y_1^2 \right)_{y_1=49}^{y_1=51} = 50, \\
 f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 \\
 &= \int_{49}^{51} \frac{1}{12} dy_1 = \frac{1}{12} (y_1)_{y_1=49}^{y_1=51} = \frac{1}{6}, \\
 E[Y_2] &= \int_2^8 y_2 f_2(y_2) dy_2 \\
 &= \int_2^8 y_2 \frac{1}{6} dy_2 = \left(\frac{1}{12} y_2^2 \right)_{y_2=2}^{y_2=8} = 5,
 \end{aligned}$$

then covariance is

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 250 - (50)(5) =$$

(choose one) (i) **0** (ii) **1** (iii) **2**.

ii. Since Y_1 and Y_2 are uncorrelated, $\text{Cov}(Y_1, Y_2) = 0$, then correlation is

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{0}{\sqrt{0.245 \times 0.245}} =$$

(choose one) (i) **0** (ii) **1** (iii) **2**.

iii. Random variables Y_1 and Y_2 are uncorrelated, $\text{Cov}(Y_1, Y_2) = 0$, because they are *independent*,

$$E(Y_1 Y_2) = E(Y_1)E(Y_2).$$

(i) **True** (ii) **False**

(b) *Machine B.* Bivariate density function for machine B is

$$f(y_1, y_2) = \begin{cases} \frac{1}{4}, & 49 \leq y_1 \leq 51, 4 \leq y_2 \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

- i. Since Y_1 and Y_2 are independent,
 $\text{Cov}(Y_1, Y_2) =$ (choose one) (i) **0** (ii) **1** (iii) **2**.
- ii. Since Y_1 and Y_2 are independent,
 correlation $\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} =$ (choose one) (i) **0** (ii) **0.5** (iii) **1**.

4. *More Covariance and Correlation.*

Consider

$$f(y_1, y_2) = \begin{cases} \frac{6}{5}(2 - y_1)(1 - y_2) & 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 1, y_1 + 2y_2 < 2 \\ 0 & \text{elsewhere} \end{cases}$$

which has a non-rectangular range.

(a) Since

$$\begin{aligned} E[Y_1 Y_2] &= \frac{6}{5} \int_0^1 \int_0^2 (y_1 y_2) (2 - y_1 - 2y_2 + y_1 y_2) dy_1 dy_2 \\ &= \frac{6}{5} \int_0^1 \int_0^2 (2y_1 y_2 - y_1^2 y_2 - 2y_1 y_2^2 + y_1^2 y_2^2) dy_1 dy_2 \\ &= \frac{6}{5} \int_0^1 \left(y_1^2 y_2 - \frac{1}{3} y_1^3 y_2 - y_1^2 y_2^2 + \frac{1}{3} y_1^2 y_2^2 \right) \Big|_{y_1=0}^{y_1=2} dy_2 \\ &= \frac{6}{5} \int_0^1 \left(4y_2 - \frac{8}{3} y_2 - 4y_2^2 + \frac{8}{3} y_2^2 \right) dy_2 \\ &= \frac{6}{5} \int_0^1 \left(\frac{4}{3} y_2 - \frac{4}{3} y_2^2 \right) dy_2 \\ &= \frac{6}{5} \left(\frac{4}{6} y_2^2 - \frac{4}{9} y_2^3 \right) \Big|_{y_2=0}^{y_2=1} = \frac{4}{15}, \\ E[Y_1] &= \int_0^2 y_1 f_1(y_1) dy_1 = \frac{6}{5} \int_0^2 y_1 \left(1 - \frac{1}{2} y_1 - \frac{1}{4} y_1^2 + \frac{1}{8} y_1^3 \right) dy_1 \\ &= \frac{6}{5} \left(\frac{1}{2} y_1^2 - \frac{1}{6} y_1^3 - \frac{1}{16} y_1^4 + \frac{1}{40} y_1^5 \right) \Big|_{y_1=0}^{y_1=2} = \frac{14}{25}, \\ E[Y_2] &= \int_0^1 y_2 f_2(y_2) dy_2 = \frac{6}{5} \int_0^1 y_2 (2 - 2y_2 - 2y_2^2 + 2y_2^3) dy_2 \\ &= \frac{6}{5} \left(y_2^2 - \frac{2}{3} y_2^3 - \frac{2}{4} y_2^4 + \frac{2}{5} y_2^5 \right) \Big|_{y_2=0}^{y_2=1} = \frac{7}{25}, \end{aligned}$$

then covariance is

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = \frac{4}{15} - \left(\frac{14}{25}\right) \left(\frac{7}{25}\right) =$$

(choose one) (i) $\frac{205}{1875}$ (ii) $\frac{206}{1875}$ (iii) $\frac{207}{1875}$.

(b) Since

$$\begin{aligned} E[Y_1^2] &= \int_0^2 y_1^2 f_1(y_1) dy_1 = \frac{6}{5} \int_0^2 y_1^2 \left(1 - \frac{1}{2}y_1 - \frac{1}{4}y_1^2 + \frac{1}{8}y_1^3\right) dy_1 \\ &= \frac{6}{5} \left(\frac{1}{3}y_1^3 - \frac{1}{8}y_1^4 - \frac{1}{20}y_1^5 + \frac{1}{48}y_1^6\right)_{y_1=0}^{y_1=2} = \frac{202}{25}, \end{aligned}$$

$$V[Y_1] = E[Y_1^2] - [E[Y_1]]^2 = \frac{202}{25} - \left(\frac{14}{25}\right)^2 = \frac{4854}{625},$$

$$\begin{aligned} E[Y_2^2] &= \int_0^1 y_2^2 f_2(y_2) dy_2 = \frac{6}{5} \int_0^1 y_2^2 (2 - 2y_2 - 2y_2^2 + 2y_2^3) dy_2 \\ &= \frac{6}{5} \left(\frac{2}{3}y_2^3 - \frac{2}{4}y_2^4 - \frac{2}{5}y_2^5 + \frac{2}{6}y_2^6\right)_{y_2=0}^{y_2=1} = \frac{3}{25}, \end{aligned}$$

$$V[Y_2] = E[Y_2^2] - [E[Y_2]]^2 = \frac{3}{25} - \left(\frac{7}{25}\right)^2 = \frac{26}{625},$$

then correlation is

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{\frac{206}{1875}}{\sqrt{\frac{4854}{625} \times \frac{26}{625}}} \approx$$

(choose one) (i) **0.19** (ii) **0.23** (iii) **0.34**.