d5 and d6 mystery PhL 8.4.13

This matter is resolved. See red notes below

The mystery is that for a large N finite pulse train, I seem to get different spectra depending on which of the two methods I use. The ensemble method gives δ_6 as I show in both old and new Section 35. But what happens in the autocorrelation method? I get this paradox: R"(z) must be positive by its definition, but my calculation gives R"(z) of both signs.

First, we know this,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\,|\,Y''(z)\,|^2 = \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \qquad (34.14)$$

$$R''(z) = \frac{T_1}{T}\,|\,Y''(z)\,|^2 = \sum_{s=-\infty}^{\infty} r_s\, z^{-s}$$

which seems to say R"(z) must be positive definite. But what is r_s ?

s=0 $\quad r_0 = \langle y_n^2\rangle_1 = \mu^2 + \sigma^2$

$s \neq 0 \quad r_s = \langle y_m y_{m+s}\rangle_1 = \mu^2 = \dfrac{1}{(2N+1)}\sum_{n=-N}^{N} y_n\, y_{n+s}$ no limit *(32.16)* (35.15)

This last expression is the problem! The expression is "correct" as stated if you understand that the quantity $y_n\, y_{n+s}$ has *implicit* θ function restrictions of this form (taken from Section 35 start)

$$y_m y_{m+s} = y_m y_{m+s}\,\theta(m \le N)\theta(m \ge -N)\,\theta(m+s \le N)\theta(m+s \ge -N)$$

$$= y_m y_{m+s}\,\theta(m \le N)\theta(m \ge -N)\,\theta(s \le N-m)\theta(s \ge -N-m) \qquad (35.1a)$$

This is that all-important gray parallelogram support region idea. These θ's restrict the n sum so in reality it is this

$$r_s = \frac{1}{2N+1}\sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} y_m y_{m+s}$$

so there is s-dependence of r_s now from the sum limits! When you then go to the ensemble level, this is

$$\langle r_s\rangle = \frac{2N+1-|s|}{2N+1}\,f(s) \qquad \text{where } f(s) = y_m y_{m+s} \text{ stationary}$$

Then when independence is added

$$f(s) = \begin{cases} \mu^2 & s \neq 0 \\ \sigma^2 + \mu^2 & s = 0 \end{cases} . \qquad \text{// stationarity and independence assumed} \qquad (35.15)$$

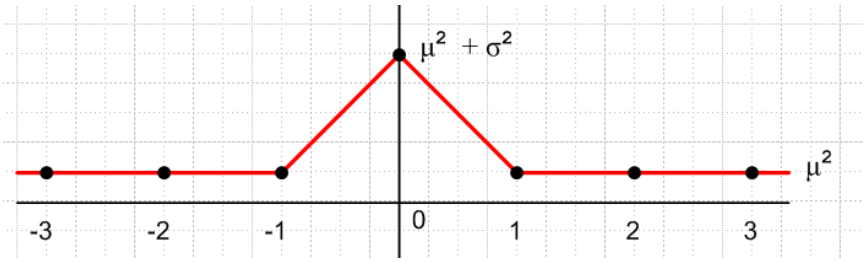
you get this

$$\langle r_s \rangle = \frac{2N+1 - |s|}{2N+1} \mu^2 \qquad s \neq 0$$

$$\langle r_0 \rangle = 1 \ (\sigma^2+\mu^2) \qquad s \neq 0$$

The first result is very different from saying $r_s = \mu^2$! You then end up with the δ_6 result, not the δ_5 result. This was a fairly subtle issue and it had me confused for several days.

I think the extent of r_s is now limited to -2N to +2N so if would be zero outside that range. Then



is the same, but it cuts off at ±2N. Then we should have

$$R''(z) = \sum_{s=-2N}^{2N} r_s z^{-s} = \sum_{s \neq 0} \{ \mu^2 \} z^{-s} + \{ \sigma^2+\mu^2 \} = \mu^2 \sum_{s=-2N}^{2N} z^{-s} + \sigma^2$$

I then call upon

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi \, \delta_5(k,N) . \qquad -\infty < k < \infty \qquad (13.3)$$

which is going to give a δ_5 result, not a δ_6 result! Then $z = e^{i\omega T_1}$ from (24.1) we have $z^{-s} = e^{-i\omega T_1 s}$, so setting $k = \omega T_1$ we get

$$\sum_{s=-2N}^{2N} z^{-s} = 2\pi \, \delta_5(\omega T_1, 2N)$$

and so

$$R''(z) = \sigma^2 + \mu^2 \, 2\pi \, \delta_5(\omega T_1, 2N)$$

Now δ_5 looks like this

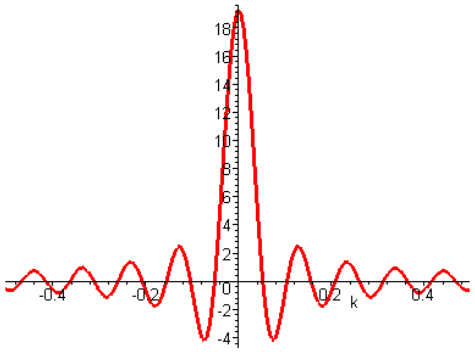
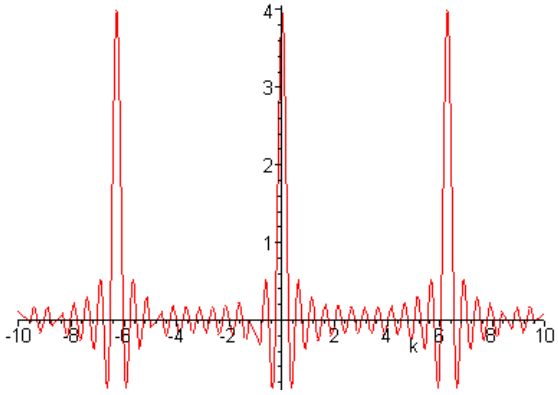


Fig A.6

and similarly from a new plot I made

```
plot(d5,k=-10..10);
```



Then if σ^2 is very small and μ^2 large, which must be possible, we have

$$R''(z) = \sigma^2 + \mu^2\, 2\pi\, \delta_5(\omega T_1, 2N)$$

and this goes negative about half the time! But I said R"(z) had to be positive all the time!

I think something is wrong about r_s.

Go back to

$$R''(z) = \sum_{s=-\infty}^{\infty} r_s\, z^{-s}$$

If you know that r_s is symmetric in s, easy to show that

$$R''(z) = \sum_{s=-\infty}^{\infty} r_s \cos(\omega T_1 s)$$

but this can have either sign! So just being symmetric in s is not enough to make R"(z) be positive! It has to be some other detail of r_s.

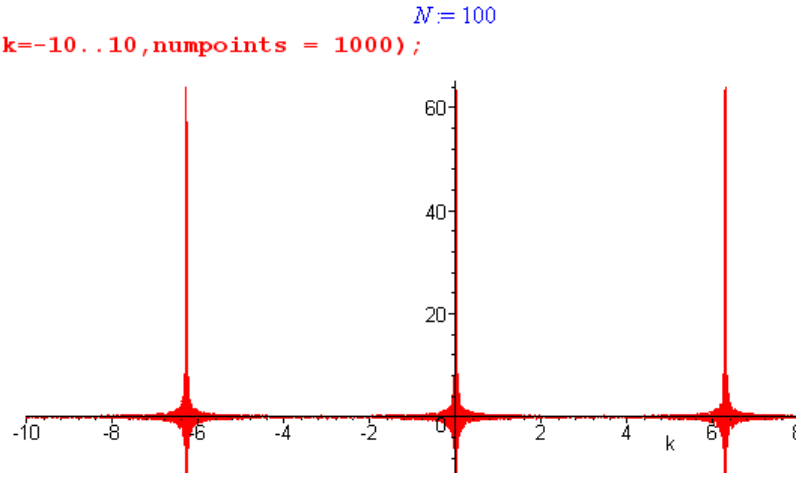
$$r_s = \sum_{m=-N}^{N} y_m y_{m+s} = \langle y_m y_{m+s} \rangle_1$$

Isn't there some tapering off at the ends? But not really. The above formula is exact, what is not exact is this assumption

$$\langle y_m y_{m+s} \rangle_1 = \langle y_m \rangle_1^2 = \mu^2$$

Except for extremely large N, there is no justification to write this! But for extremely large N, we will find that δ_5 and δ_6 give the same result and the negativity of δ_5 becomes less and less

```
N := 100;
plot(d5,k=-10..10,numpoints = 1000);
```



So I guess that is the resolution of this mystery.

(1) for finite N, we cannot say $\langle y_m y_{m+s} \rangle_1 = \langle y_m \rangle_1^2$ even though we can say $\langle y_m y_{m+s} \rangle = \langle y_m \rangle^2$ for independent

Comment: If we blindly repeat the finite N calculation using the autocorrelation method with the same r_s shown in *** , we obtain

$$R"(z) = \sum_{s=-2N}^{2N} r_s z^{-s} = \sum_{s \neq 0} \{ \mu^2 \} z^{-s} + \{ \sigma^2 + \mu^2 \} = \mu^2 \sum_{s=-2N}^{2N} z^{-s} + \sigma^2$$

$$= \sigma^2 + \mu^2 \, 2\pi \, \delta_5(\omega T_1, 2N)$$

which does give the correct result as $N \to \infty$. But for finite N, as Fig A.6 shows, if σ^2 is small and μ^2 large, then this function R"(z) goes *negative* in places near the peaks, which conflicts with $R"(z) = \frac{T_1}{T} | Y"(z) |^2$

having to be positive definite (δ_6 never goes negative). This paradox is explained by realizing that for finite N, if the ensemble random variables are independent, we might have for $s \neq 0$ that $< y_m y_{m+s} > = < y_m >^2$, but the identifications of (35.9) are only approximate for finite N so $< y_m y_{m+s} >_1 = < y_m >_1^2$ is only approximately true, allowing the error just noted.

OK , this matter is now resolved and I entered a the above as Comment 2 at the end of the Section 35 section on the finite calculation.