New FT Section 32 (f) PhL 3.26.05

This is a new section at the end of Section 32, so no existing equation numbers get altered. We just tack this on at the end. But as I read now, I think <...> should be $<....>_1$. !!! Yes, in FT I have fixed that up. This is all good. So this doc is just where I developed this new section 32 (f).

(f) Z Transform Wiener-Khintchine theorem for a Pulse Train

The Wiener-Khintchine theorem was stated above in section (c) as

$$R_x(\omega) = |X(\omega)|^2 \qquad (32.8)$$

where $R_x(\omega)$ is the Fourier Integral transform of the autocorrelation function $r_x(t)$

$$r_x(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \ . \qquad (32.1)$$

From the amplitudes y_n of an infinite pulse train one can define an **autocorrelation sequence** in analogy with the autocorrelation function,

$$r_s \equiv \lim_{N\to\infty} \left[\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s}\right] = \langle y_n\, y_{n+s}\rangle \ . \qquad (32.16)$$

Although $r_x(t)$ (our particular definition) has no normalization factor, we have added $\frac{1}{(2N+1)}$ to the definition of r_s in order to obtain the finite result $r_s = \langle y_n\, y_{n+s}\rangle$.

It is convenient to use this shorthand notation for r_s,

$$r_s = \frac{1}{(2N+1)} \sum_{n=-\infty}^{\infty} y_n\, y_{n+s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n\, y_{n+s} \qquad (32.17)$$

where $T = (2N+1)T_1$ is the duration of the pulse train. We know that this infinite T is going to cancel another T in any "application" so we allow it to exist temporarily, as in (33.22) where $T = [2\pi\delta(0)]T_1$.

The Z transform of r_s is given by

$$R''(z) \equiv \sum_{s=-\infty}^{\infty} r_s\, z^{-s} = \sum_{s=-\infty}^{\infty} \left\{ \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n\, y_{n+s} \right\} z^{-s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} [y_n\, z^n]\, [y_{n+s}\, z^{-(n+s)}]$$

$$= \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [y_n\, z^n]\, [y_m\, z^{-m}] \qquad m \equiv n+2$$

$$= \frac{T_1}{T} \left[\sum_{n=-\infty}^{\infty} y_n z^n \right] \left[\sum_{m=-\infty}^{\infty} y_m z^{-m} \right] = \frac{T_1}{T} Y''(z)^* Y''(z)$$

so we have obtained this Z Transform version of the Wiener-Khintchine theorem, which we compare to the regular version,

$$R''(z) = \frac{T_1}{T} |Y''(z)|^2 \qquad \text{Z Transform Wiener-Khintchine} \qquad (32.18)$$

$$R_x(\omega) = |X(\omega)|^2 \ . \qquad \text{regular Wiener-Khintchine} \qquad (32.8)$$

Here $X(\omega)$ is the Fourier Integral Transform of the pulse train $x(t)$,

$$x(t) = \sum_{n=-\infty}^{\infty} y_n x_{pulse}(t - t_n). \qquad (25.1)$$

while $Y''(z)$ is the Z transform of the sequence of pulse train amplitudes.

We now jump ahead in our presentation in order to show a key use of $R''(z)$. Below we will define the spectral power and energy density $\mathcal{P}(\omega)$ and $\mathcal{E}(\omega)$ of an infinite pulse train and its pulse as

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad (33.23)$$

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \ . \qquad (33.24)$$

The regular Wiener-Khintchine theorem (32.8) that $R_x(\omega) = |X(\omega)|^2$ then tell us

$$\mathcal{P}(\omega) = \frac{R_x(\omega)}{2\pi T} \qquad \text{or} \qquad \mathcal{E}(\omega) = \frac{R_x(\omega)}{2\pi} \ . \qquad (34.3)$$

Later we will show that

$$\mathcal{P}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.14)$$

$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} |Y''(z)|^2 \ . \qquad (32.19)$$

Inserting the Z transform Wiener-Khintchine theorem (32.18) then gives

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) R''(z) \qquad (32.20)$$

which provides a simple relation between $\mathcal{P}(\omega)$ and $R''(z)$.