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## 35. Statistical Pulse Trains

### (a) What is a Statistical Pulse Train?

Suppose we use a logic analyzer or digital scope to capture a 1 GHz PAM signal (zeros and ones) for 1 μsec so a sequence of K = 1,000 symbols $y_m$ are stored. We could compute the average value for this sequence and we might find $\langle y_m \rangle_1 = 0.55$. Here $\langle ... \rangle_1$ refers to a **horizontal average** over a single pulse train amplitude sequence. By its definition, this average cannot depend on the index m, we are just computing an average of the $y_m$ values in the sequence,

$$\langle y_m \rangle_1 = (1/K) \Sigma_{m=1}^{K} y_m \ .$$

Now it might be that during this 1 μsec, there was something unusual about the data being sent, so that this measurement of $\langle y_m \rangle_1 = 0.55$ is not really representative over a longer term. One alternative would be to capture a much longer sequence. Rather than do this, our approach will be to capture very many 1 μsec pulse trains and form an **ensemble** of these representative pulse trains. Perhaps we do this for a whole minute, so the ensemble then has a huge number (call it I) of pulse trains. We then write down these amplitude sequences in a vertical list on a very long piece of paper, one sequence below the next. We then number the positions in the sequences 1 to N and we then define $\langle y_m \rangle$ as the **vertical average** through this ensemble of the sequences in position m. This vertical average is written

$$\langle y_m \rangle = (1/I) \Sigma_{i=1}^{I} y_m^{(i)}$$

and in theory this could be different for different columns m. For example, consider this ensemble of sequences which has K = 6 and I = 3:

| 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|
| a | b | c | d | e | f | sequence #1 |
| a' | b' | c' | d' | e' | f' | sequence #2 |
| a" | b" | c" | d" | d" | f" | sequence #3 |

Ensemble Fig 35.1

We would have

$\langle y_m \rangle_1 = (a+b+c+d+e+f)/6.$ // horizontal average for the first sequence
$\langle y_3 \rangle = (c + c' + c")/3$ // vertical average for column 3

Our pulse train sequence has a **random variable** $Y_m$ associated with each pulse train position, m = 1,2...K, and the vertical average $\langle y_m \rangle$ is the expected value of $Y_m$ over the ensemble, normally written $\langle y_m \rangle = E(Y_m)$. The experiment associated with $Y_m$ is the generation of a pulse train by some Apparatus, the experiment is run many times, and the outcomes for random variable $Y_m$ are the values of the symbols located in position m of these pulse trains. See Appendix G for the meaning of "random variable".

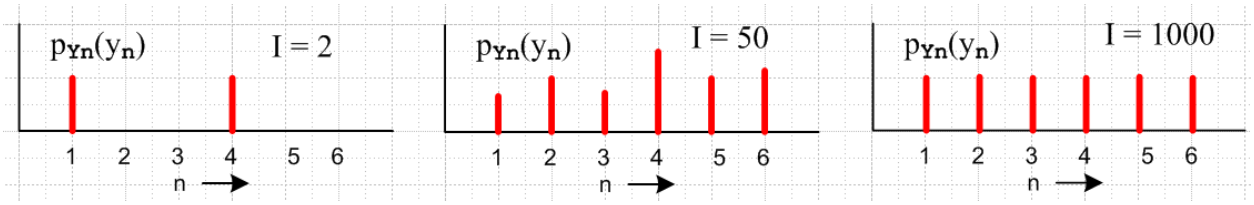
The ensemble is characterized by various statistical properties, such as $E(Y_m)$ or $E(Y_n Y_m Y_k)$, each being a vertical average down through the ensemble. Our **statistical pulse train** is an idealized pulse train

whose amplitudes form a **statistical sequence** whose statistics ( like $E(Y_m)$ and $E(Y_nY_mY_k)$ ) exactly match those of the ensemble. It is not any particular member of the ensemble, since any member might deviate somehow. And it is certainly not the average of the pulse trains in the ensemble, since this average pulse train would have amplitudes $y_m = \langle y_m \rangle$ which has nothing to do with anything ( and also would have illegal symbols, e.g., $y_m = 1/2$). If K were very large, a pulse train of the ensemble might come close to being a statistical pulse train.

Common usage refers to our statistical pulse train as a **random pulse train** whose sequence of amplitudes is a **random sequence**. The problem with this terminology is that the word "random" suggests for example that $\langle y_m \rangle$ = the mean value of $Y_m$ in the ensemble = 1/2. That is to say, the word random suggests a flat distribution where p(1) = 1/2 and p(0) = 1/2. But if p(1) = 1/3 and p(0) = 2/3, our ensemble would still be associated with a statistical pulse train, Any value of p ≡ p(1) would describe a statistical pulse train, along with the other statistical properties. This same subtle implication of the word random appears in Appendix G on "random variables" and basic probability theory.

In what follows, we shall only be concerned with the first and second order statistics of the statistical pulse train, which are $\langle y_m \rangle = E(Y_m)$ and $\langle y_m y_n \rangle = E(Y_m Y_n)$.

How many member pulse trains must the ensemble contain? Could it have only two pulse trains? The answer is that the ensemble could be small, but then the statistical properties of the random variables as discovered by the ensemble would have a large amount of error. For example, for column n there is some normalized distribution $p_{Yn}(y_n)$ associated with the random variable $Y_n$. Suppose $y_n$ took values from {1,2,3,4,5,6}, and suppose $p_{Yn}(y_n)$ were a flat distribution. Here might be the different views of $p_{Yn}(y_n)$ as obtained from ensembles with I = 2, 50 and 1000 pulse trains :




Since $E(Y_n) = \Sigma_n\, y_n\; p_{Yn}(y_n)$, the error propagates into this and all other expectation values associated with the statistical pulse train. For any required degree of accuracy δ of the statistical quantities, we can find some value $I_d$ such that for $I > I_\delta$ the ensemble will realize that accuracy. In theory, we just imagine that I = ∞, and then everything is perfect.

**(b) The region of support for $y_m y_{m+s}$ for a finite pulse train**

The product of amplitudes $y_m y_{m+s}$ appears many times below. For a finite pulse train it is useful to see where the product vanishes and where it does not vanish in terms of m and s. In the previous section we discussed a finite pulse train having $y_n$ with n = 1 to K, but now we instead have $y_n$ with n = -N to N so our finite pulses train now have (2N+1) amplitudes $y_n$. We imagine the finite pulse train embedded in an infinite pulse train that has all zeros to the left of -N and to the right of +N. In this case we may write for this infinite pulse train,

$$y_m y_{m+s} = y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(m+s \le N)\theta(m+s \ge -N)$$

$$= y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(s \le N-m)\theta(s \ge -N-m)$$

where we use an inequality-style Heaviside step function θ. The product of the four θ functions and thus the quantity $y_m y_{m+s}$ vanishes outside the gray parallelogram in this drawing of the (m,s) plane,

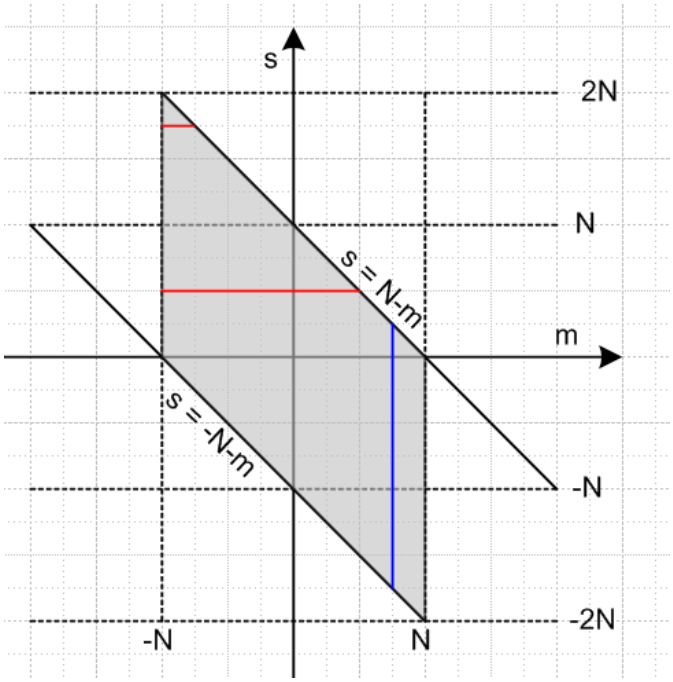



Fig 35.2

Thus, assuming m is in range -N to N, we may write

$$y_m y_{m+s} = \begin{cases} y_m y_{m+s} & \text{for } -N-m \le s \le N-m \\ 0 & \text{for } s < -N-m \text{ or } s > N-m \end{cases} \tag{35.1}$$

Below we shall be interested in the following quantity (the autocorrelation sequence),

$$r_s = \langle y_m y_{m+s} \rangle_1 = \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s} \qquad \text{// a horizontal average} \tag{35.2}$$

$$= \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(s \le N-m)\theta(s \ge -N-m)$$

$$= \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(m \le N-s)\theta(m \ge -N-s)$$

$$= \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} y_m y_{m+s} \quad . \tag{35.3}$$

The range of m appearing in this sum is represented by a red line in the figure for two positive values of s. For s > 2N there is no range left so $r_s = 0$, and the same when s < -2N. Thus, we may regard

$$r_s = r_s\,\theta(s\le 2N)\theta(s\ge -2N)\ . \qquad (35.4)$$

Another quantity of interest, R"(z) = Z Transform of $r_s$, therefore has this restricted sum range for a finite pulse train,

$$R"(z) = \sum_{s=-\infty}^{\infty} r_s\, z^{-s} = \sum_{s=-2N}^{2N} r_s\, z^{-s}\ . \qquad (35.5)$$

Notice that $\langle y_m y_{m+s}\rangle_1 = r_s$ does not depend on m due to its "horizontal" average definition shown in (35.2). The non-dependence of $\langle y_m y_{m+s}\rangle_1$ on m has nothing to do with "stationarity" as will be discussed below. That is, $\langle y_m y_{m+s}\rangle_1$ is for a particular pulse train, it is not an ensemble average.

**(c) The Spectral Power Density for a specific pulse train**

There are *many ways* to write the power density expression for a pulse train, see (32.18) and (34.14a). For an infinite pulse train we may write (reader should perhaps ponder each form),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s}\rangle_1\, z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} r_s\, z^{-s}$$

$$= \mathcal{P}_{pulse}(\omega)\, R"(z) \quad = \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\,|\,Y"(z)\,|^2 \qquad r_s = \frac{T_1}{T}\sum_{m=-\infty}^{\infty} y_m y_{m+s}$$

$$= \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\,\Big(\sum_{m=-\infty}^{\infty} y_m\, z^{-m}\Big)^*\Big(\sum_{n=-\infty}^{\infty} y_n\, z^{-n}\Big)$$

$$= \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\,\Big(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} y_m\, y_n\, z^{m-n}\Big) \qquad // \ z = e^{i\omega T_1}\ \text{so}\ (z^{-m})^* = z^m$$

$$= \mathcal{P}_{pulse}(\omega)\frac{T_1}{T}\,\Big(\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} y_m\, y_{m+s}\, z^{-s}\Big) \qquad \mathcal{P}(\omega)\ \text{INFINITE} \qquad (35.6)$$

Recall that T is the duration of the infinite pulse train, written as $T = 2\pi\delta(0)$ in (33.22). This T always cancels away, so we are not concerned about its infinite nature. It is a limit $(2N+1)T_1$ as $N\to\infty$.

For a finite pulse train we find, using (35.3) and (35.5) and Fig 35.2,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s}\rangle_1\, z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} r_s\, z^{-s}$$

$$= \mathcal{P}_{pulse}(\omega)\, R"(z) \quad = \mathcal{P}_{pulse}(\omega)\frac{1}{2N+1}\,|\,Y"(z)\,|^2 \qquad r_s = \frac{1}{2N+1}\sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} y_m y_{m+s}$$

$$= \mathcal{P}_{pulse}(\omega)\frac{1}{2N+1}\,\Big(\sum_{m=-N}^{N} y_m\, z^{-m}\Big)^*\Big(\sum_{n=-N}^{N} y_n\, z^{-n}\Big)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\Big(\sum_{m=-N}^{N}\sum_{n=-N}^{N} y_m\,y_n\,z^{\mathbf{m-n}}\Big)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\Big(\sum_{m=-N}^{N}\sum_{s=-N-m}^{N-m} y_m\,y_{\mathbf{m+s}}\,z^{\mathbf{-s}}\Big) \qquad \mathcal{P}(\omega)\ \text{FINITE} \qquad (35.7)$$

In the last expression, the sum on s is represented by the blue line in Fig 35.2. In all these expressions, one can regard $y_{\mathbf{m}}y_{\mathbf{m+s}}$ and $y_{\mathbf{m}}\,y_{\mathbf{n}}$ as being stripped of their Heaviside θ functions as in (35.3). That is to say, there are no places in any of the above expressions where $y_{\mathbf{m}}y_{\mathbf{m+s}} = 0$ because it is out of range. Later when we add stationarity, we will replace $y_{\mathbf{m}}\,y_{\mathbf{m+s}}$ by f(s) which is never 0 in an out-of-range sense.

**(d) The mean Spectral Power Density for an Ensemble of pulse trains**

We now apply our ensemble average <....> to the previous equation groups to get for the infinite case,

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-\infty}^{\infty}\langle\langle y_{\mathbf{m}}y_{\mathbf{m+s}}\rangle_{\mathbf{1}}\rangle\,z^{\mathbf{-s}} = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-\infty}^{\infty}\langle r_{\mathbf{s}}\rangle\,z^{\mathbf{-s}}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\langle R''(z)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_{\mathbf{1}}}{T}\langle|\,Y''(z)\,|^{\mathbf{2}}\rangle \qquad \langle r_{\mathbf{s}}\rangle = \frac{T_{\mathbf{1}}}{T}\sum_{m=-\infty}^{\infty}\langle y_{\mathbf{m}}y_{\mathbf{m+s}}\rangle$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_{\mathbf{1}}}{T}\,\Big\langle\Big(\sum_{m=-\infty}^{\infty} y_{\mathbf{m}}\,z^{\mathbf{-m}}\Big)^{*}\Big(\sum_{n=-\infty}^{\infty} y_{\mathbf{n}}\,z^{\mathbf{-n}}\Big)\Big\rangle$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_{\mathbf{1}}}{T}\,\Big(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\langle y_{\mathbf{m}}\,y_{\mathbf{n}}\rangle\,z^{\mathbf{m-n}}\Big)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_{\mathbf{1}}}{T}\,\Big(\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\langle y_{\mathbf{m}}\,y_{\mathbf{m+s}}\rangle\,z^{\mathbf{-s}}\Big) \qquad \langle\mathcal{P}(\omega)\rangle\ \text{INFINITE} \qquad (35.8)$$

and for an ensemble of finite pulse trains,

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-2N}^{2N}\langle\langle y_{\mathbf{m}}y_{\mathbf{m+s}}\rangle_{\mathbf{1}}\rangle z^{\mathbf{-s}} = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-2N}^{2N}\langle r_{\mathbf{s}}\rangle\,z^{\mathbf{-s}}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\langle R''(z)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\langle|\,Y''(z)\,|^{\mathbf{2}}\rangle, \quad \langle r_{\mathbf{s}}\rangle = \frac{1}{2N+1}\sum_{m=\max(-N,-N-s)}^{\min(N,N-s)}\langle y_{\mathbf{m}}y_{\mathbf{m+s}}\rangle$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\Big\langle\Big(\sum_{m=-N}^{N} y_{\mathbf{m}}\,z^{\mathbf{-m}}\Big)^{*}\Big(\sum_{n=-N}^{N} y_{\mathbf{n}}\,z^{\mathbf{-n}}\Big)\Big\rangle$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\Big(\sum_{m=-N}^{N}\sum_{n=-N}^{N}\langle y_{\mathbf{m}}\,y_{\mathbf{n}}\rangle\;\;z^{\mathbf{m-n}}\Big)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{1}{2N+1}\,\Big(\sum_{m=-N}^{N}\sum_{s=-N-m}^{N-m}\langle y_{\mathbf{m}}\,y_{\mathbf{m+s}}\rangle\,z^{\mathbf{-s}}\Big) \qquad \langle\mathcal{P}(\omega)\rangle\ \text{FINITE} \qquad (35.9)$$

**(e) Adding Stationarity to the Ensemble Situation**

The stationarity assumption is that the Apparatus generating the pulse trains of the ensemble is a "**stationary** stochastic process" which means (at least) that

$\langle y_m \rangle$ does not depend on m // where $\langle y_m \rangle$ does not vanish
$\langle y_m y_{m+s} \rangle$ does not depend on m // where $\langle y_m y_{m+s} \rangle$ does not vanish (35.10)

Recall that for a finite pulse train, $\langle y_m \rangle$ vanishes outside the range (-N,N) while $\langle y_m y_{m+s} \rangle$ vanishes outside the gray region in Fig 35.2. For an infinite pulse train these objects don't vanish anywhere.

From Appendix G we know that

$$\text{cov}(X,Y) = E(XY) - \mu_x \mu_y \qquad \textit{(G.24b)}$$
$$\Rightarrow \text{cov}(Y_m, Y_{m+s}) = \langle y_m y_{m+s} \rangle - \langle y_m \rangle \langle y_{m+s} \rangle = \langle y_m y_{m+s} \rangle - \langle y_m \rangle^2 \quad \text{// stationarity of } \langle y_m \rangle$$

Assuming $\langle y_m \rangle$ and $\langle y_m y_{m+s} \rangle$ are stationary is the same as assuming $\langle y_m \rangle$ and cov $(Y_m, Y_{m+s})$ are stationary. If we make no stationarity requirement on other statistical measures (like $\langle y_m^3 \rangle$), this limited sense of stationarity is usually called **wide-sense stationarity** (WSS).

This stationarity assumption applies to both finite and infinite pulse trains. It is an assumption about the process which generates pulse trains regardless of their length.

It is helpful to define $\mu \equiv \langle y_m \rangle$ and $\langle y_m y_{m+s} \rangle = f(s)$ and rewrite the stationarity assumption in this manner for a finite pulse train, where the second equation invokes Fig 35.2's gray region,

$$\langle y_m \rangle = \begin{cases} \mu & \text{for } -N \le m \le N \\ 0 & \text{for } m < -N \text{ or } m > N \end{cases}$$

$$\langle y_m y_{m+s} \rangle = \begin{cases} f(s) & \text{for } -N-m \le s \le N-m \\ 0 & \text{for } s < -N-m \text{ or } s > N-m \end{cases} \qquad \text{// m in (-N,N)} \qquad (35.11)$$

For an infinite pulse train, the upper lines apply and then $\langle y_m \rangle = \mu$ and $\langle y_m y_{m+s} \rangle = f(s)$ everywhere.

With this stationarity assumption, we can write

$$\langle r_s \rangle = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle = \frac{T_1}{T} f(s) \sum_{m=-\infty}^{\infty} [1] = f(s) \qquad \text{// infinite pulse train}$$

$$\langle r_s \rangle = \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} \langle y_m y_{m+s} \rangle = \frac{1}{2N+1} f(s) \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} [1] =$$
$$= \frac{1}{2N+1} f(s) (2N+1 - |s|) \qquad \text{// see red lines in Fig 35.2}$$

$$= \frac{2N+1 - |s|}{2N+1} f(s) \qquad \text{// finite pulse train}$$

To summarize :

$$\langle r_s \rangle = f(s) \qquad \text{// infinite pulse train}$$
$$\langle r_s \rangle = \frac{2N+1 - |s|}{2N+1} f(s) \qquad \text{// finite pulse train} \tag{35.12}$$

Here then are the infinite pulse train expressions assuming stationarity :

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle\langle y_m y_{m+s} \rangle_1 \rangle z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle r_s \rangle z^{-s}$$
$$= \mathcal{P}_{pulse}(\omega) \langle R''(z) \rangle = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \langle | Y''(z) |^2 \rangle \qquad \langle r_s \rangle = f(s)$$
$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \langle \left( \sum_{m=-\infty}^{\infty} y_m z^{-m} \right)^* \left( \sum_{n=-\infty}^{\infty} y_n z^{-n} \right) \rangle$$
$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle y_m y_n \rangle z^{m-n} \right)$$
$$= \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} f(s) z^{-s} \qquad \langle \mathcal{P}(\omega) \rangle \text{ INFINITE (stat)} \tag{35.13}$$

In the last line we used

$$\frac{T_1}{T} \left( \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \right) = \frac{T_1}{T} \left( \sum_{s=-\infty}^{\infty} f(s) z^{-s} \right) \sum_{m=-\infty}^{\infty} [1] = \sum_{s=-\infty}^{\infty} f(s) z^{-s}$$

and then the last line just replicates the first line given that $\langle r_s \rangle = f(s)$.

And here are the *finite* pulse train $\langle \mathcal{P}(\omega) \rangle$ expressions assuming stationarity :

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} \langle\langle y_m y_{m+s} \rangle_1 \rangle z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} \langle r_s \rangle z^{-s}$$
$$= \mathcal{P}_{pulse}(\omega) \langle R''(z) \rangle = \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \langle | Y''(z) |^2 \rangle \qquad \langle r_s \rangle = \frac{2N+1 - |s|}{2N+1} f(s)$$
$$= \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \langle \left( \sum_{m=-N}^{N} y_m z^{-m} \right)^* \left( \sum_{n=-N}^{N} y_n z^{-n} \right) \rangle$$
$$= \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \left( \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle y_m y_n \rangle z^{m-n} \right)$$
$$= \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} f(s) z^{-s} \frac{2N+1 - |s|}{2N+1} \qquad \langle \mathcal{P}(\omega) \rangle \text{ FINITE (stat)} \tag{35.14}$$

In the last line we did this manipulation on the last line of (35.9) (see the gray region in Fig 35.2)

$$\frac{1}{2N+1}\sum_{m=-N}^{N}\sum_{s=-N-m}^{N-m}\langle y_m\, y_{m+s}\rangle\, z^{-s} \;=\; \frac{1}{2N+1}\sum_{s=-2N}^{2N}\sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} f(s)\, z^{-s}$$

$$= \frac{1}{2N+1}\sum_{s=-2N}^{2N} f(s)\, z^{-s} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)}[1] = \frac{1}{2N+1}\sum_{s=-2N}^{2N} f(s)\, z^{-s}\ (\,2N+1-|s|\,)$$

One again, we see that the final expression in (35.14) replicates the first one with $\langle r_s\rangle$ as shown.

**(f) Adding Independence along with Stationarity to the Ensemble Situation**

As shown in Appendix G, if $Y_n$ and $Y_m$ are independent random variables, then for $s \neq 0$,

$f(s) = \langle y_m\, y_{m+s}\rangle = E(Y_n Y_{m+s}) = E(Y_n)E(Y_{m+s}) = \langle y_m\rangle\langle y_{m+s}\rangle = \langle y_m\rangle\langle y_m\rangle = \langle y_m\rangle^2$ // stationarity

or

$f(s) = \mu^2 \qquad \mu = \langle y_m\rangle$

But for s = 0, the result is different. We write

$$f(0) = \langle y_m y_m\rangle = \langle y_m^2\rangle = \sigma^2 + \mu^2 \qquad \text{where } \sigma^2 \equiv \langle y_m^2\rangle - \langle y_m\rangle^2 = \langle y_m^2\rangle - \mu^2 \qquad \textit{(G.30)}$$

To summarize:

$$f(s) = \begin{cases} \mu^2 & s \neq 0 \\ \sigma^2 + \mu^2 & s = 0 \end{cases} . \qquad \text{// stationarity and independence assumed} \qquad (35.15)$$

For the last time (!), we consider the various $\langle \mathcal{P}(\omega)\rangle$ expressions now with stationarity and independence assumed. Rather than write all the expressions, we focus just on the first equation (same as the last) of the group (35.13) for the infinite case,

$$\langle \mathcal{P}(\omega)\rangle = \mathcal{P}_{pulse}(\omega)\sum_{s=-\infty}^{\infty}\langle r_s\rangle\, z^{-s} = \mathcal{P}_{pulse}(\omega)\sum_{s=-\infty}^{\infty} f(s)\, z^{-s}$$

$$= \mathcal{P}_{pulse}(\omega)\,[\, f(0) + \sum_{s\neq 0} f(s)\, z^{-s}\,]$$

$$= \mathcal{P}_{pulse}(\omega)\,[(\sigma^2+\mu^2) + \mu^2 \sum_{s\neq 0} z^{-s}\,]$$

$$= \mathcal{P}_{pulse}(\omega)\,[\sigma^2 + \mu^2 \sum_{s=-\infty}^{\infty} z^{-s}\,] \qquad \text{// now use (13.2) with } z = e^{i\omega T_1} \text{ to get :}$$

$$= \mathcal{P}_{pulse}(\omega)\,[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\,] \;. \qquad \langle \mathcal{P}(\omega)\rangle \text{ INFINITE (stat+indep)} \qquad (35.16)$$

We shall return to this classic result below. Meanwhile, we have from the first equation (same as the last) of the group (35.14) for the *finite* case,

$$\begin{aligned}
\langle \mathcal{P}(\omega) \rangle &= \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-2N}^{2N} f(s)\, z^{-s}\, \frac{2N+1-|s|}{2N+1} \\
&= \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, f(0) + \sum_{s\neq 0} f(s)\, z^{-s}\, \frac{2N+1-|s|}{2N+1} \,\Big] \\
&= \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, (\sigma^2+\mu^2) + \mu^2 \sum_{s\neq 0} z^{-s}\, \frac{2N+1-|s|}{2N+1} \,\Big] \\
&= \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, \sigma^2 + \mu^2 \sum_{s=-2N}^{2N} z^{-s}\, \frac{2N+1-|s|}{2N+1} \,\Big] \qquad \text{// see App. D and comments below} \\
&= \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, \sigma^2 + \mu^2\, \frac{1}{2N+1}\, \frac{\sin^2[(N+1/2)\omega T_1]}{\sin^2(\omega T_1/2)} \,\Big] \qquad \text{// now use (A.20) to get:} \\
&= \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, \sigma^2 + \mu^2\, 2\pi\, \delta_6(\omega T_1, N) \,\Big] \ . \qquad \langle \mathcal{P}(\omega) \rangle \text{ FINITE (stat+indep)} \qquad (35.17)
\end{aligned}$$

The evaluation of the $\Sigma_s$ in the third last line is non-trivial and is carried out in Appendix D. The last line expresses the second last line in the language of Appendix A where $\delta_6$ is the delta function model shown in (A.20). From (A.21) we know that

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad \textit{(A.21)}$$

so the last line of (35.17) becomes in the limit $N\to\infty$,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\Big[\, \sigma^2 + \mu^2\, 2\pi \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \,\Big]$$

which (as expected) agrees with the last line of (35.16) for the infinite case.

Evaluation of the $\Sigma_s$ in (3.17) is not really necessary since we can compute $\langle \mathcal{P}(\omega) \rangle$ using a different equation from the group (35.13), namely

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\, \frac{1}{2N+1} \Big( \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle y_m\, y_n \rangle\ z^{m-n} \Big) \ . \qquad (35.18)$$

Sometimes double sums (or double integrals) are easier to evaluate than a single-sum representation of the same function. The double sum in (35.18) may be evaluated as follows, using (35.15),

$$\sum_{n=-N}^{N}\sum_{m=-N}^{N} \langle y_m y_n \rangle z^{n-m} = \sum_{n=-N}^{N}\left[\sum_{m=n} f(0)\, z^{n-m} + \sum_{m\neq n} f(n-m)\, z^{n-m}\right]$$

$$= \sum_{n=-N}^{N}\left[(\sigma^2+\mu^2) + \mu^2 \sum_{m\neq n} z^{n-m} + \right] \qquad \text{// in first term } \sum_{m=n} z^{n-m} = 1$$

$$= \sum_{n=-N}^{N}\left[\,\sigma^2 + \mu^2 \sum_{m=-N}^{N} z^{n-m} + \right]$$

$$= (2N+1)\,\sigma^2 + \mu^2 \left(\sum_{n=-N}^{N} z^n\right)\left(\sum_{m=-N}^{N} z^m\right)^*$$

$$= (2N+1)\,\sigma^2 + \mu^2 \left[\, 2\pi\, \delta_5(\omega T_1, N)\,\right]^2 \qquad \text{// from (13.3) used twice with } k = \omega T_1$$

$$= (2N+1)\,\sigma^2 + \mu^2 \left[\, (2N+1)\, 2\pi\, \delta_6(\omega T_1, N)\,\right] \qquad \text{// from (A.20)} \qquad (35.19)$$

and therefore

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega)\, \frac{1}{2N+1} \left\{ (2N+1)\,\sigma^2 + \mu^2 \left[ (2N+1)\, 2\pi\, \delta_6(\omega T_1, N) \right] \right\}$$

$$= \mathcal{P}_{pulse}(\omega)\left[\,\sigma^2 + \mu^2\, 2\pi\, \delta_6(\omega T_1, N)\right]$$

which agrees with the last line of (35.17), and we bypassed doing the Appendix D sum.

**(g) Comparison between the two kinds of averages <..> and <...>$_1$**

We now want to compare the two averages appearing in the above equations :

$\langle y_m \rangle_1$ cannot depend on m from its definition as a horizontal average over a sequence.
$\langle y_m \rangle$ might depend on m, but does not with the stationarity assumption

$\langle y_m y_{m+s} \rangle_1 = r_s$ cannot depend on m from its definition (32.17), but generally depends on s
$\langle y_m y_{m+s} \rangle$ may depend on m as well as s, but with stationarity depends only on s

$\langle y_m y_{m+s} \rangle_1$ is not associated with any random variables
$\langle y_m y_{m+s} \rangle$ is associated with random variables $Y_m$ and $Y_{m+s}$ as outlined in section (a) above

If $Y_m$ and $Y_{m+s}$ are independent random variables, then $\langle y_m y_{m+s} \rangle = \langle y_m \rangle \langle y_{m+s} \rangle$ (see App G).
In contrast, the statement $\langle y_m y_{m+s} \rangle_1 = \langle y_m \rangle_1 \langle y_{m+s} \rangle_1 = \langle y_m \rangle_1 \langle y_m \rangle_1 = \langle y_m \rangle_1^2$ might be true, but has no connection to any random variables upon whose independence this factoring could be postulated.

(35.20)

In order to make a connection between these two kinds of averages, we have to make some assumptions.

For an ensemble of pulse trains of *very long* length (large N), we assume that each ensemble member is a "statistical pulse train" in that it very closely has the statistics of the ensemble as a whole. This does not mean that the ensemble pulse trains are the same.

One implication of this assumption is that all member pulse trains (i = 1,2.. I) then have the same autocorrelation function and we may then write

$$r_s^{(i)} = \langle y_m y_{m+s} \rangle_1^{(i)} = (1/N) \Sigma_{m=1}^{N} y_m^{(i)} y_{m+s}^{(i)} = \text{the same for all i = 1 to I.}$$

$$\equiv r_s \equiv \langle y_m y_{m+s} \rangle_1 \quad . \tag{35.21}$$

Consider then,

$$\Sigma_{m=1}^{N} \langle y_m y_{m+s} \rangle = N \langle y_m y_{m+s} \rangle \quad . \qquad \text{// stationarity}$$

On the left side of this equation we insert the definition of $\langle y_m y_{m+s} \rangle$,

$$\langle y_m y_{m+s} \rangle = (1/I) \Sigma_{i=1}^{I} y_m^{(i)} y_{m+s}^{(i)} \ , \tag{35.22}$$

with this result

$$\Sigma_{m=1}^{N} (1/I) \Sigma_{i=1}^{I} y_m^{(i)} y_{m+s}^{(i)} = N \langle y_m y_{m+s} \rangle \ .$$

Now reorder the sums to get,

$$(N/I) \Sigma_{i=1}^{I} [(1/N) \Sigma_{m=1}^{N} y_m^{(i)} y_{m+s}^{(i)}] = N \langle y_m y_{m+s} \rangle$$

$$(N/I) \Sigma_{i=1}^{I} \langle y_m y_{m+s} \rangle_1^{(i)} = N \langle y_m y_{m+s} \rangle$$

$$(1/I) \Sigma_{i=1}^{I} \langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle \qquad \text{// use (35.21) and cancel the N's}$$

$$\langle y_m y_{m+s} \rangle_1 [ (1/I) \Sigma_{i=1}^{I} 1 ] = \langle y_m y_{m+s} \rangle$$

$$\langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle \ . \qquad \text{// horizontal average equals vertical average}$$

As a special case, we can set s = 0 to get

$$\langle y_m^2 \rangle_1 = \langle y_m^2 \rangle \ .$$

In similar fashion, our very large N assumption implies that all pulse trains in the ensemble have the same horizontal mean, so

$$\langle y_m \rangle_1^{(i)} \equiv \langle y_m \rangle_1 \ . \tag{35.23}$$

Repeating the above steps now for $\langle y_m \rangle$,

$\Sigma_{m=1}^{N} \langle y_m \rangle = N \langle y_m \rangle$ // stationarity

$\Sigma_{m=1}^{N} [ (1/I) \Sigma_{i=1}^{I} y_m^{(i)} ] = N \langle y_m \rangle$

$(N/I) \Sigma_{i=1}^{I} (1/N) \Sigma_{m=1}^{N} [y_m^{(i)}] = N \langle y_m \rangle$

$(N/I) \Sigma_{i=1}^{I} \langle y_m \rangle_1^{(i)} = N \langle y_m \rangle$

$(1/I) \Sigma_{i=1}^{I} \langle y_m \rangle_1 = \langle y_m \rangle$ // use (35.23) and cancel the N's

$\langle y_m \rangle_1 = \langle y_m \rangle$ .

Thus, assuming stationarity and very large N, we have found that

$\langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle$
$\langle y_m y_m \rangle_1 = \langle y_m y_m \rangle$
$\langle y_m \rangle_1 = \langle y_m \rangle$ . // all three lines only for very large N (35.24)

These relations are equalities for $N = \infty$, are approximately valid for large N, and are invalid for small N.

If we add to our assumptions so far that $Y_m$ and $Y_{m+s}$ are independent, not only may we write

$E(Y_m Y_{m+s}) = E(Y_m)E(Y_{m+s}) \quad \Leftrightarrow \quad \langle y_m y_{m+s} \rangle = \langle y_m \rangle \langle y_{m+s} \rangle$ // independent, $s \neq 0$

but we may also write, using stationarity and independence

$\langle y_m y_{m+s} \rangle = \langle y_m \rangle \langle y_m \rangle = \langle y_m \rangle^2$ $s \neq 0$

which by (35.24) then justifies the claim sometimes made that,

$\langle y_m y_{m+s} \rangle_1 = \langle y_m \rangle_1 \langle y_m \rangle_1 = \langle y_m \rangle_1^2$ $s \neq 0$ //stationary + independent + $N \to \infty$ (35.25)

**(h) Miscellaneous topics**

**The Two Pathways for computing $\mathcal{P}(\omega)$**

From (35.6) and (35.8) we have obtained these two general "formulas" for the spectral power density of a an infinite pulse train, prior to assuming stationarity or independence. We first write these as

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left( \sum_{s=-\infty}^{\infty} r_s z^{-s} \right)$$ first line of (35.6) "autocorrelation"

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left( \sum_{m=-\infty}^{\infty} y_m z^{-m} \right)^* \left( \sum_{n=-\infty}^{\infty} y_n z^{-n} \right) \quad (35.26)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\frac{T_1}{T}\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} y_m\, y_n\, z^{m-n}\right) \qquad \text{third line of (35.6) "double sum"}$$

The corresponding equations for finite pulse trains are,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\left(\sum_{s=-2N}^{2N} r_s\, z^{-s}\right) \qquad \text{first line of (35.7) "autocorrelation"}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_1}{T}\left(\sum_{m=-N}^{N} y_m\, z^{-m}\right)^*\left(\sum_{n=-N}^{N} y_n\, z^{-n}\right) \qquad (35.26)'$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\frac{T_1}{T}\left(\sum_{m=-N}^{N}\sum_{n=-N}^{N} y_m\, y_n\, z^{m-n}\right) \qquad \text{third line of (35.7) "double sum"}$$

In either case we have two distinct pathways for the computation of $\mathcal{P}(\omega)$. We might call these pathways the **autocorrelation approach** which uses $r_s$, and the **double-sum approach** which directly uses the $y_n$ data without the intermediary of $r_s$. We saw earlier that the double-sum approach provided an easier pathway for the finite pulse train case, leading to the last line of (35.17) using the steps shown in (35.19), but in general the autocorrelation approach is the one most commonly used.

The autocorrelation approach will be used in Section 37 for the AMI line code, in Section 38 for the Change/Hold Line code, and in Appendix F for infinite pulse trains with repeated subsequences such as the Maximum Length Sequence output by a shift register generator.

**The equivalence of $\mathcal{P}(\omega)$ and $\langle\mathcal{P}(\omega)\rangle$ for infinite pulse trains**

We now rewrite (35.26), this time using an ensemble average on the lower equation,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\left(\sum_{s=-\infty}^{\infty}\langle y_m y_{m+s}\rangle_1\, z^{-s}\right) \qquad \text{one pulse train} \qquad (35.6)$$

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_1}{T}\left(\sum_{s=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\langle y_m\, y_{m+s}\rangle\, z^{-s}\right) \qquad \text{ensemble of pulse trains} \qquad (35.8)$$

When stationarity is assumed for the process creating the pulse trains, in the second line above we may slide $\langle y_m\, y_{m+s}\rangle\, z^{-s}$ to the left through the $\Sigma_m$ sum, which sum becomes $\Sigma_m[1] = T/T_1$ and we then have this pair of formulas

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-\infty}^{\infty}\langle y_m y_{m+s}\rangle_1\, z^{-s} = \mathcal{P}_{\mathbf{pulse}}(\omega)\left(\sum_{s=-\infty}^{\infty} r_s\, z^{-s}\right) \qquad \text{one pulse train} \qquad (35.7)$$

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\sum_{s=-\infty}^{\infty}\langle y_m\, y_{m+s}\rangle\, z^{-s} \quad . \qquad \text{ensemble of pulse trains} \qquad (35.8)$$

(35.27)

Our point is merely that, according to (35.24) which states $\langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle$, the two right hand side expressions are the *same* for infinite pulse trains, and <u>there is no distinction</u> between $\langle \mathcal{P}(\omega) \rangle$ for the ensemble and $\mathcal{P}(\omega)$ for any member of the ensemble. There is also no distinction between $r_s$ and $\langle r_s \rangle$ since every pulse train in the ensemble has the same autocorrelation function as in (35.21).

**Facts about Infinite Uncorrelated Statistical Pulse Trains**

Here we use the word "uncorrelated" as a sort of euphemism for what is really a stationary and independent statistical pulse train as described above. It is true that independent does imply uncorrelated as shown in Appendix G (c). Since the converse is not true, our term is somewhat inaccurate.

In the special case of stationarity and independence, we found that from (35.12),

$$r_s = \langle r_s \rangle = f(s) = \begin{cases} \mu^2 & s \neq 0 \\ \sigma^2 + \mu^2 & s = 0 \end{cases} \qquad \langle y_m \rangle = \mu, \quad \langle y_m^2 \rangle = \sigma^2 + \mu^2 \qquad (35.12)$$

where

$$r_s = \lim_{N \to \infty} \left[ \frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n \, y_{n+s} \right] . \qquad (32.16)$$

A plot of $r_s$ is usually presented in this traditional manner,

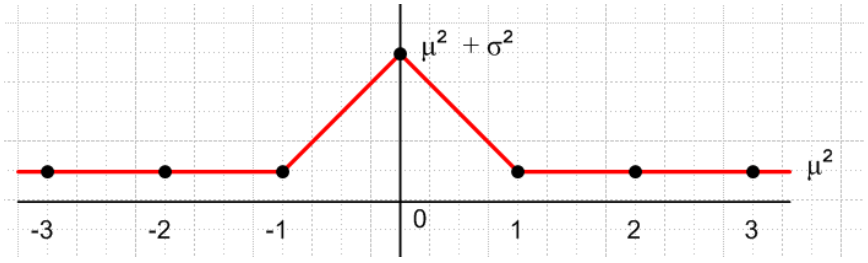



Fig 35.3

and the corresponding ensemble average power spectrum was shown in the last line of (35.16)

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \left[ \sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right] . \qquad (35.16)$$

The spectrum has a continuous piece $\sigma^2 \, \mathcal{P}_{pulse}(\omega)$ proportional to the variance $\sigma^2$ of the $y_n$ over the ensemble, and a discrete set of lines proportional to the square of the mean $\mu = \langle y_m \rangle$ of the ensemble. According to the comments above and (35.24), we can think of $\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$ of any *member* of the ensemble, $\mu = \langle y_m \rangle = \langle y_m \rangle_1$ as the mean for that member, and similarly $\sigma^2 = \langle y_m^2 \rangle - \mu^2 = \sigma^2 = \langle y_m^2 \rangle_1 - \mu^2$ as the variance for that member of the ensemble. With this reinterpretation of $\mu^2$ and $\sigma^2$ as properties of a single pulse train, we then have for any member of the ensemble,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[ \sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right] \quad . \tag{35.28}$$

Since

$$2\pi\delta(\omega T_1 - 2\pi m) = (2\pi/T_1)\, \delta(\omega - 2\pi m/T_1) = \omega_1\, \delta(\omega - m\omega_1) = \delta\left(\frac{\omega}{\omega_1} - m\right) \tag{35.29}$$

we can write these alternate forms for (35.28) :

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \omega_1 \mu^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \right] \tag{35.28a}$$

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{\mathbf{pulse}}(\omega) + \omega_1 \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1)\delta(\omega - m\omega_1) \tag{35.28b}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} \delta\left(\frac{\omega}{\omega_1} - m\right) \right] \tag{35.28c}$$

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{\mathbf{pulse}}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1)\, \delta\left(\frac{\omega}{\omega_1} - m\right) \tag{35.28d}$$

<u>Verification of (35.28)</u> // "trust but verify"

To find external verification for (35.28), we write the expression in terms of f where $\omega = 2\pi f$,

$$\mathcal{P}_{\mathbf{pulse}}(\omega) = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} = \frac{|\mathcal{X}_{\mathbf{pulse}}(f)|^2}{2\pi T_1} \qquad \text{// (34.4) and text after (1.4)}$$

$$P(f) = 2\pi\mathcal{P}(\omega) \qquad \text{// (34.4) last item}$$

$$2\pi\, \delta(\omega T_1 - 2\pi m) = 2\pi\, \delta(2\pi f T_1 - 2\pi m) = (1/T_1)\, \delta(f - m/T_1) \quad .$$

Then (35.28) becomes

$$\frac{1}{2\pi} P(f) = \frac{|\mathcal{X}_{\mathbf{pulse}}(f)|^2}{2\pi T_1} \left( \sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(f - m/T_1) \right)$$

or

$$P(f) = \frac{|\mathcal{X}_{\mathbf{pulse}}(f)|^2}{T_1} \left( \sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(f - m/T_1) \right) \quad . \tag{35.28e}$$

This may be compared to result (A.17) in Appendix A of Xiong which we quote,

$$\Psi_s(f) = \frac{|G(f)|^2}{T} \left( \sigma_a^2 + \frac{m_a^2}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) \right) \tag{A.17}$$

**QED**

**Specialization for symbols in the set {A,B}**

As a simple example, suppose the set of symbols that our $y_i$ can take is just {A,B}. Then the pmf distribution associated with random variable $Y_n$ (see Appendix G) is quite simple :

$$p_{Y_n}(x) = p \qquad \text{if } x = A$$
$$p_{Y_n}(x) = 1\text{-}p \qquad \text{if } x = B \ . \tag{35.30}$$

We can then compute

$$\mu = \langle y_m \rangle = [p]A + [1\text{-}p]B \ . \tag{35.31}$$

For $n \neq m$ we then have

$$\langle y_m y_n \rangle = \langle y_m \rangle^2 = \mu^2 = \{ pA + (1\text{-}p)B \}^2$$

$$= [pp]\, AA + [p(1\text{-}p)]\, AB + [(1\text{-}p)p]BA + [(1\text{-}p)(1\text{-}p)]BB \tag{35.32}$$

and for n = m,

$$\langle y_m^2 \rangle = [p]AA + [(1\text{-}p)]\, BB \ . \tag{35.33}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. Notice how the long expression for $\langle y_m y_n \rangle$ makes complete sense if one enumerates all possibilities. The variance may be computed as

$$\sigma^2 = \langle y_n^2 \rangle - \mu^2 = [p]AA + [(1\text{-}p)]\, BB - \{ pA + (1\text{-}p)B \}^2$$

$$= [p(1\text{-}p)]\, (A\text{-}B)^2 \tag{35.34}$$

where Maple helps out,

```
p*A^2 + (1-p)*B^2 - ( p*A + (1-p)*B)^2: factor(%);
```

$$-p\,(A-B)^2\,(-1+p)$$

.

With the reinterpretations of $\mu^2$ and $\sigma^2$ noted above (now applying to a single pulse train), we then have for our infinite statistical pulse train with independent amplitudes in the set {A,B}

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, [\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \tag{35.28}$$

$$= \mathcal{P}_{pulse}(\omega)\, [\ p(1\text{-}p)(A\text{-}B)^2 + (pA + (1\text{-}p)B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \quad . \tag{35.35}$$

For the interesting case that p = 1/2 this becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [\ \frac{1}{4}\ (A\text{-}B)^2 + \frac{1}{4}\ (A+B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \qquad (35.36)$$

For the classic unipolar case A = 1 and B = 0, both coefficients are 1/4. For bipolar with A = 1 and B = -1 the coefficients are 1 and 0. Since the mean is then $\mu = 0$, the discrete spectrum goes away. Since for both amplitudes $|y_m| = 1$, we are not surprised to find $\sigma^2 = 1$.

**(i) Statistical Uncorrelated Pulse Trains: Summary and Example**

Here then is a brief summary of the above results:

**Spectral Power Density of an Uncorrelated Statistical Pulse Train** (35.37)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\ x_{\mathtt{pulse}}(t - nT_1) \qquad \text{// or } \sum_{n=-N}^{N} \text{ for a finite pulse train}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [\ \sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \qquad \text{// infinite} \qquad \textit{(35.28)}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [\ \sigma^2 + \mu^2\ 2\pi\ \delta_6(\omega T_1, N)\ ] \qquad \text{// finite} \qquad \textit{(35.17)}$$

If symbols are restricted to {A,B} then :

$\sigma^2 = p(1-p)\ (A-B)^2$ and for p = 1/2 $\sigma^2 = \frac{1}{4}\ (A-B)^2$ *(35.34)*

$\mu = pA + (1-p)B$ and for p = 1/2 $\mu^2 = \frac{1}{4}\ (A+B)^2$ *(35.31)*

Example: Suppose A = 1 and B = 0 so that $\sigma^2 = p(1-p)$ and $\mu^2 = p^2$. If we go on to assume p = 1, then every pulse has amplitude 1 and we have a simple pulse train. In this case $\sigma^2 = 0$ and $\mu^2 = 1$ so

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{\mathtt{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules}$$

which agrees with our infinite simple pulse train result (33.25). As we reduce p below p = 1, the line spectra are scaled down by $\mu^2 = p^2 < 1$, and a continuous spectrum starts to appear with $\sigma^2 = p(1-p)$. The randomness (variance $\sigma^2$) of the amplitude magnitudes creates a continuous component in the spectral power density. As p reaches p = 1/2, we get $\sigma^2 = 1/4$ and $\mu^2 = 1/4$ to give this classic result

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\ [(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m)\ ] \qquad p = 1/2 \qquad (35.38)$$

Finally, when p reaches p = 0, then every pulse has B = 0, $x(t) \equiv 0$, $\sigma^2 = 0$, $\mu^2 = 0$, and so $\mathcal{P}(\omega) = 0$. There is nothing left.

**(j) A numerical example of a Statistical Pulse Train**

Recall from box (34.4) that for a finite pulse train,

$$\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T} = \frac{|X(\omega)|^2}{2\pi(2N+1)T_1} \qquad \mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} . \tag{35.39}$$

Therefore we can write the last line of (35.17) as,

$$\frac{\langle|X(\omega)|^2\rangle}{(2N+1)} = |X_{pulse}(\omega)|^2 \left[(1/4) + (1/4)\, 2\pi\, \delta_6(\omega T_1, N)\right] . \tag{35.40}$$

We shall use for $x_{pulse}(t)$ a square pulse of height A=1 and width $\tau = T_1 = 1$ so that, from (9.2),

$$| X_{pulse}(\omega) | = \text{sinc}(\omega/2) . \tag{35.41}$$

Since our pulse train will be fairly short (N = 20 pulses) and since we shall only average a small number of pulse trains (M = 10), we know our result will not exactly match (35.40). Still, we hope to see in our result some kind of continuous background spectrum which approximates the curve $(1/4)|X_{pulse}(\omega)|^2 = (1/4)\,\text{sinc}^2(\omega/2)$, and we expect to see delta-function-like peaks which, since $2\pi\delta_6(0,N) = (2N+1)$, have a peak value of about $(1/4)41 = 10.25$. Since this will be added to the continuous background, the central peak should have a height of $10.25 + .25 = 10.5$. However, for our small ensemble, we won't have exactly p = 1/2, so the delta peak won't be exactly 10.5 units high.

We know that $\delta_6$ has identical peaks spaced by $2\pi$, but we expect the non-central peaks to be suppressed by the $\text{sinc}^2(\omega/2)$ zeros which occur at $\omega = m(2\pi)$.

Here is a self-documented Maple program which generates $\langle|X(\omega)|^2\rangle$. The program also generates the quantity $\langle X(\omega)\rangle$ upon which we shall comment in section (k) below.

```
restart; assume(w,real); with(plots):
T1 is pulse duration, N is number of pulses in each pulse train
T1 := 1: N := 20:
Spectrum for standard box shaped pulse
Xpulse := T1*sin(w*T1/2)/(w*T1/2):
Subsequent calls to r() will generate either 0 or 1 with prob p = 1/2
r := rand(0..1):
```

```
Clear averaging accumulators. M is number of pulse trains to average
Xas means X absolute-value squared
X_acc := 0:Xas_acc := 0: M := 10:
for m from 1 to M do
  Create a random pulse train
   for n from -N to N do y[n] := r(); od:
   Compute the Digital Fourier Transform Y'(w) of the pulse train from (23.5)
   Yp := T1*sum(y[j]*exp(-I*w*j*T1),j=-N..N):
   Compute X(w) for the pulse train from (25.4), omitting Xpulse(w)
   X := (1/T1)* Yp;
   Accumulate both X(w) and |X(w)|^2, divide by M later for ensemble average
   X_acc := X_acc + X;
   Xas_acc := Xas_acc + eval(abs(X)^2);
od:
Compute <X(w)> and <|X(w)|^2> from the accumulators, but Xpulse not yet added
X_av_    :=  X_acc/M:
Xas_av_ := Xas_acc/M:
```

At this point, before $X_{\mathtt{pulse}}(\omega)$ is added to the result, we plot $\langle|X(\omega)|^2\rangle$. As expected, we see the peaks of $\delta_6$ spaced by $2\pi$ and having height around 10 units,

```
Plot <|X(w)|^2>/(2N+1) of (35.40) without the pulse shape
plot(Xas_av_/(2*N+1), w=-10..10,color=[red,blue],view=0..12,numpoints=500);
```

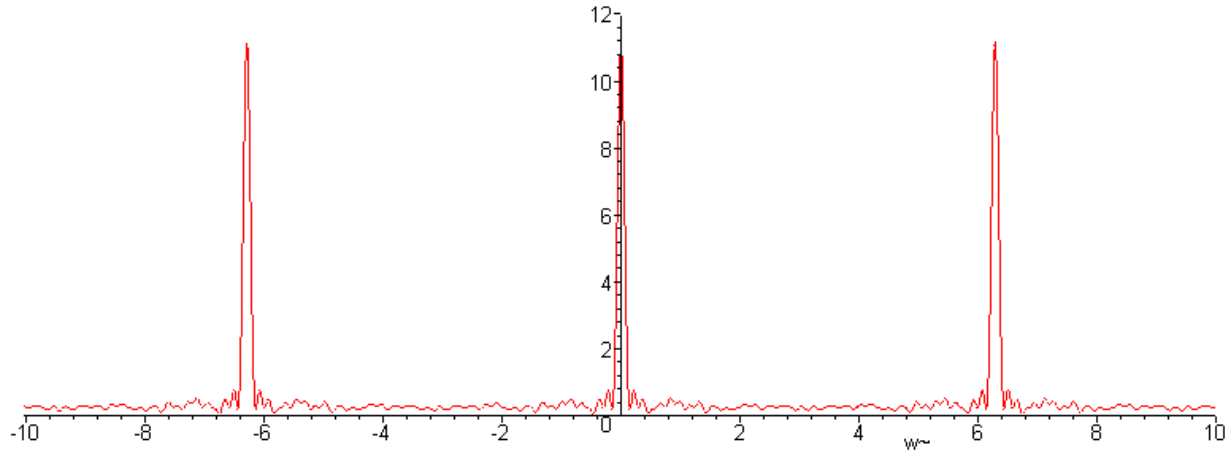



Fig 35.4

We now insert copies of $X_{\mathtt{pulse}}(\omega)$ as appropriate,

```
Install the pulse shape to get the true expressions for <X(w)> and of <|X(w)|^2>
X_av := Xpulse*X_av_:
Xas_av := Xpulse^2*Xas_av_:
```

and then we can plot $\dfrac{\langle|X(\omega)|^2\rangle}{(2N+1)}$ for $\omega$ in the same range (-10,10)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..12 ,color =
[red,blue],numpoints = 500);
```

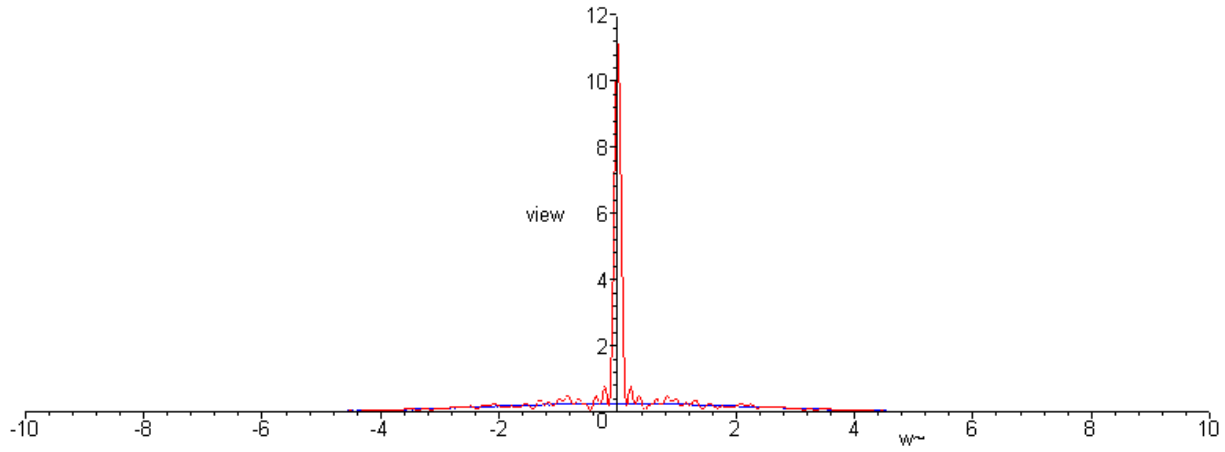



Fig 35.5

We see that the zeros of the sinc function have killed off the adjacent peaks.

Next, we restrict the plot height to be 0.8 units to view the detail, chopping off the central $\delta_6$ peak,

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..0.8 ,color =
[red,blue],numpoints = 500);
```

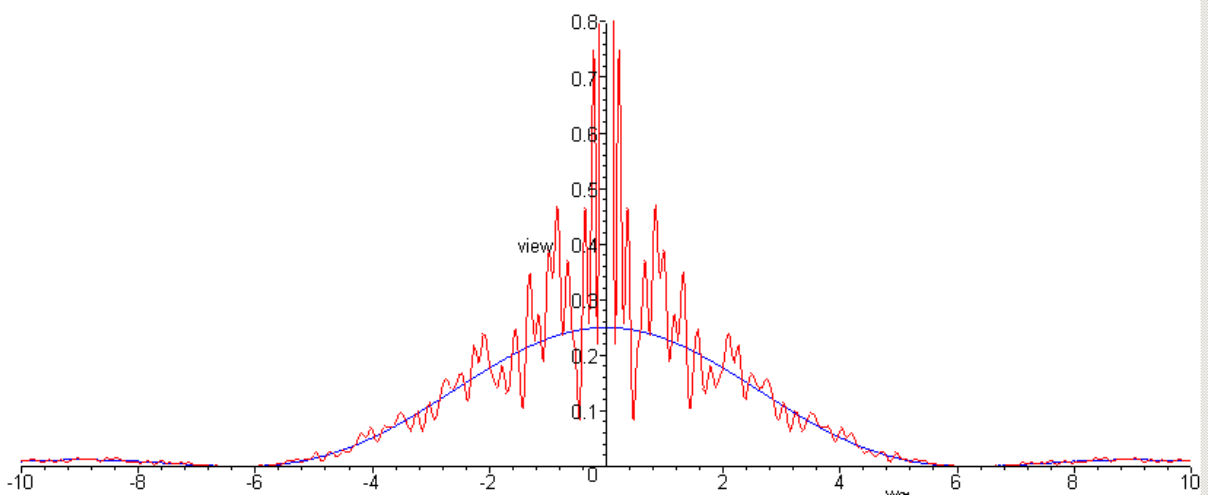



Fig 35.6

The spectrum $\frac{\langle|X(\omega)|^2\rangle}{(2N+1)}$ is seen to have a continuous component which very well approximates one quarter of the $\text{sinc}^2$ curve, as we hoped it would. This tracking also occurs away from the central peak. Here is a blow-up of the above plot for $\omega$ in the range (5, 30)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=5..30, view=0..0.02 ,color =
[red,blue],numpoints = 500);
```

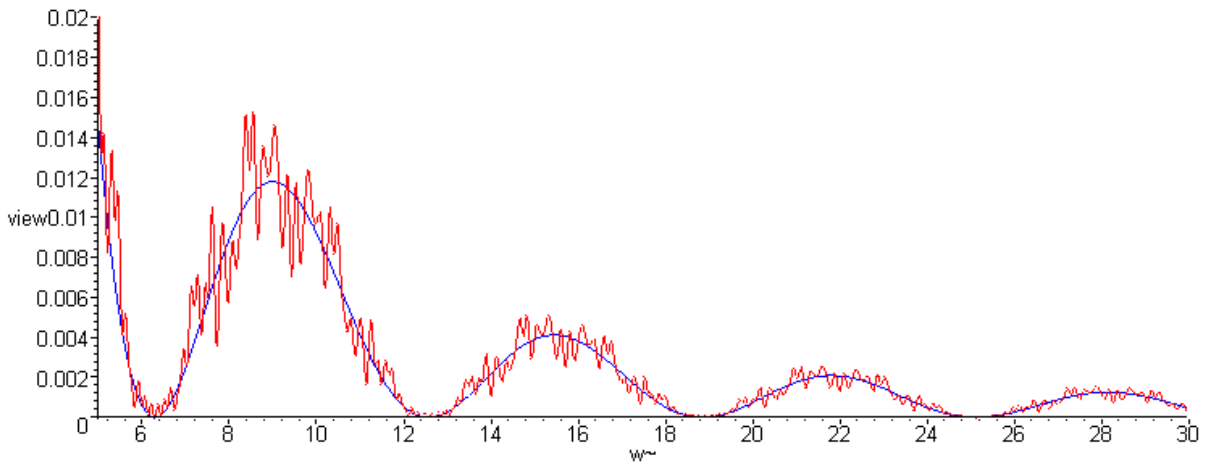



Fig 35.7

It might be noted that Maple does this work analytically, so that Xas-av is a function of ω having a large number of trigonometric terms. For the reader's interest, we show Xas-av(ω) for a typical program run :

$$Was_av := \frac{2}{5}\sin\left(\frac{1}{2}w\right)^2 ((2\cos(8w) + 2\cos(10w) + \cos(9w) + \cos(12w) + 2\cos(11w) + \cos(14w) + \cos(17w) + 2\cos(5w) + \cos(4w) + 2\cos(3w)$$
$$+ \cos(w) + \cos(2w) + 2\cos(20w) + 2\cos(18w) + \cos(19w) + \cos(15w) + \cos(13w))^2$$
$$+ (\sin(9w) + \sin(12w) + \sin(14w) + \sin(17w) - \sin(4w) + \sin(w) + \sin(2w) - \sin(19w) - \sin(15w) - \sin(13w))^2 + (1 + 2\cos(8w) + \cos(10w)$$
$$+ 2\cos(11w) + 2\cos(13w) + 2\cos(14w) + 2\cos(16w) + \cos(6w) + 2\cos(5w) + 2\cos(4w) + \cos(2w) + \cos(20w) + 2\cos(19w) + \cos(17w) + \cos(9w)$$
$$+ \cos(7w))^2 + (\sin(10w) - \sin(6w) - \sin(2w) - \sin(20w) - \sin(17w) - \sin(9w) - \sin(7w))^2 + (1 + 2\cos(7w) + \cos(9w) + \cos(13w) + \cos(14w)$$
$$+ 2\cos(15w) + \cos(8w) + 2\cos(6w) + 2\cos(5w) + \cos(4w) + 2\cos(3w) + 2\cos(2w) + 2\cos(w) + \cos(20w) + \cos(18w) + \cos(12w) + \cos(19w))^2$$
$$+ (\sin(9w) + \sin(13w) + \sin(14w) - \sin(8w) - \sin(4w) - \sin(20w) - \sin(18w) - \sin(12w) + \sin(19w))^2 + (1 + 2\cos(10w) + \cos(13w) + 2\cos(14w)$$
$$+ 2\cos(15w) + 2\cos(16w) + \cos(17w) + \cos(8w) + 2\cos(5w) + \cos(2w) + 2\cos(w) + \cos(6w) + 2\cos(18w) + \cos(19w) + \cos(9w) + \cos(7w)$$
$$+ \cos(20w))^2 + (\sin(13w) + \sin(17w) - \sin(8w) - \sin(2w) + \sin(6w) - \sin(19w) - \sin(9w) - \sin(7w) + \sin(20w))^2 + (\cos(7w) + \cos(10w) + \cos(9w)$$
$$+ 2\cos(12w) + 2\cos(13w) + \cos(15w) + \cos(5w) + 2\cos(3w) + \cos(w) + \cos(4w) + \cos(6w) + 2\cos(19w) + \cos(11w) + \cos(20w))^2$$
$$+ (\sin(7w) + \sin(10w) + \sin(9w) + \sin(15w) - \sin(5w) - \sin(w) + \sin(4w) + \sin(6w) - \sin(11w) + \sin(20w))^2 + (1 + 2\cos(7w) + \cos(12w) + \cos(13w)$$
$$+ \cos(14w) + 2\cos(15w) + \cos(16w) + \cos(8w) + 2\cos(5w) + 2\cos(3w) + 2\cos(w) + \cos(2w) + \cos(4w) + \cos(20w) + \cos(17w) + \cos(9w)$$
$$+ \cos(18w))^2 + (\sin(12w) + \sin(13w) + \sin(14w) + \sin(16w) - \sin(8w) + \sin(2w) + \sin(4w) - \sin(20w) - \sin(17w) - \sin(9w) + \sin(18w))^2 + (1$$
$$+ 2\cos(7w) + \cos(12w) + 2\cos(11w) + \cos(14w) + 2\cos(17w) + \cos(8w) + 2\cos(6w) + \cos(5w) + \cos(4w) + 2\cos(3w) + \cos(2w) + \cos(w)$$
$$+ \cos(20w) + \cos(16w) + \cos(9w))^2 + (\sin(12w) + \sin(14w) - \sin(8w) - \sin(5w) - \sin(4w) - \sin(2w) + \sin(w) - \sin(20w) - \sin(16w) - \sin(9w))^2 + (1$$
$$+ 2\cos(7w) + \cos(8w) + 2\cos(12w) + 2\cos(13w) + \cos(14w) + 2\cos(17w) + 2\cos(6w) + \cos(5w) + \cos(4w) + \cos(3w) + 2\cos(w) + 2\cos(18w)$$
$$+ \cos(11w) + \cos(10w))^2 + (\sin(8w) + \sin(14w) - \sin(5w) - \sin(4w) - \sin(3w) - \sin(11w) - \sin(10w))^2$$
$$+ (\cos(8w) + 2\cos(10w) + \cos(13w) + 2\cos(6w) + \cos(5w) + \cos(w) + \cos(2w) + \cos(3w) + \cos(20w) + \cos(17w) + \cos(12w) + \cos(9w) + \cos(7w))^2$$
$$+ (\sin(8w) + \sin(13w) - \sin(5w) - \sin(w) + \sin(2w) + \sin(3w) - \sin(20w) - \sin(17w) - \sin(12w) - \sin(9w) - \sin(7w))^2 + (\cos(8w) + \cos(10w)$$
$$+ 2\cos(14w) + \cos(16w) + \cos(17w) + \cos(6w) + \cos(5w) + 2\cos(4w) + \cos(3w) + \cos(w) + \cos(20w) + 2\cos(18w) + \cos(19w) + \cos(12w)$$
$$+ \cos(7w))^2 + (\sin(8w) + \sin(10w) + \sin(16w) + \sin(17w) - \sin(6w) - \sin(5w) - \sin(3w) + \sin(w) - \sin(20w) - \sin(19w) - \sin(12w) - \sin(7w))^2) \Big/ w^2$$

In more serious work with larger numbers, one would of course do this in a more numeric fashion, but we are able to confirm the basic results even with this small experiment.

**(k) A paradox and its resolution**

Now while here, we can back up and apply our statistical average directly to the spectrum in (34.6) using (34.7). The result is

$$<X(\omega)> = X_{\mathbf{pulse}}(\omega) \sum_{n=-\infty}^{\infty} <y_n> e^{-in\omega T_1} \quad . \qquad (34.6,7) \qquad (35.42)$$

From (35.31) we have

$$\langle y_n \rangle = [p]\, A + [1-p]\, B \qquad = p \qquad \text{if } A=1,\ B=0 \quad . \tag{35.43}$$

In this case, we can extract p from the above sum, which then collapses to form the usual (13.2) delta function sum. The result is then the same as the regular pulse train result (14.4) with an overall factor of p out front, namely,

$$\langle X(\omega) \rangle \;\; = \; p \sum_{m=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\; 2\pi\; \delta(\,\omega T_1 - 2\pi m) \;. \tag{35.44}$$

This says that our average spectrum is 100% *discrete*, there is no continuous part! If p = 1/2, the average spectrum is just 1/2 times our discrete unit amplitude pulse train spectrum (14.4). How can this be true, if we just showed in (35.28) that the average spectral density $\langle |X(\omega)|^2 \rangle$ has a *continuous* spectral component?

The answer lies in the fact that $\langle ab \rangle \neq \langle a \rangle\langle b \rangle$, where $\langle\ \rangle$ is our averaging operation. Thus

$$\langle |X(\omega)|^2 \rangle \;\neq\; \langle X(\omega) \rangle \langle X(\omega)^* \rangle \;. \tag{35.45}$$

In particular we have from the second last line of (35.9) and (35.42),

$$\sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle y_m\, y_n \rangle\; z^{m-n} \;\neq\; \Big( \sum_{n=-N}^{N} \langle y_n \rangle\, z^{-n} \Big) \Big( \sum_{m=-N}^{N} \langle y_m \rangle\, z^{m} \Big) \,. \tag{35.46}$$

These double sums *would* be equal if $\langle y_m\, y_n \rangle = \langle y_n \rangle \langle y_m \rangle$ for all m and n, but this is not true when n = m. The left sum in that case sees $\langle y_m^2 \rangle = \sigma^2 + \mu^2$ while the right side sees $\langle y_m \rangle^2 = \mu^2$.

In general we do not expect the average of a product to be the product of the averages,

$$\Big( \frac{1}{N} \Sigma_{i=1}^{N} A_i B_i \Big) \neq \Big( \frac{1}{N} \Sigma_{i=1}^{N} A_i \Big) \Big( \frac{1}{N} \Sigma_{i=1}^{N} B_i \Big) \;.$$

In the numerical example presented in the previous section we computed $\langle X(\omega) \rangle$ for a small ensemble of pulse trains. Here are plots of the real and imaginary part of $\langle X(\omega) \rangle$,

```
plot([Re(X_av),Im(X_av)], w=-10..10, view =-6..22 ,color = [red,blue],numpoints = 500);
```

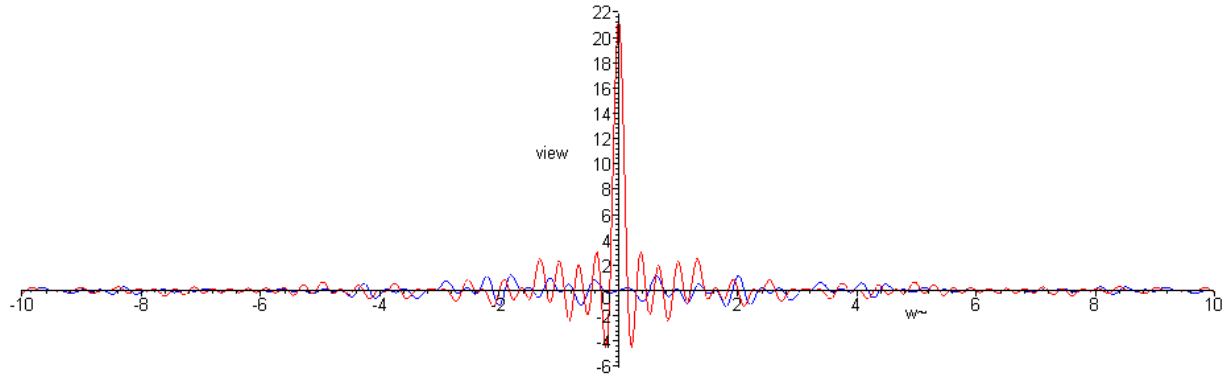



Fig 35.8

We can see that, apart from the noise of our low statistics, there is only the central peak and no continuous spectrum component. A blow-up of the central region follows,

```
plot([Re(X_av),Im(X_av)], w=-4..4, view =-6..8 ,color = [red,blue],numpoints = 500);
```

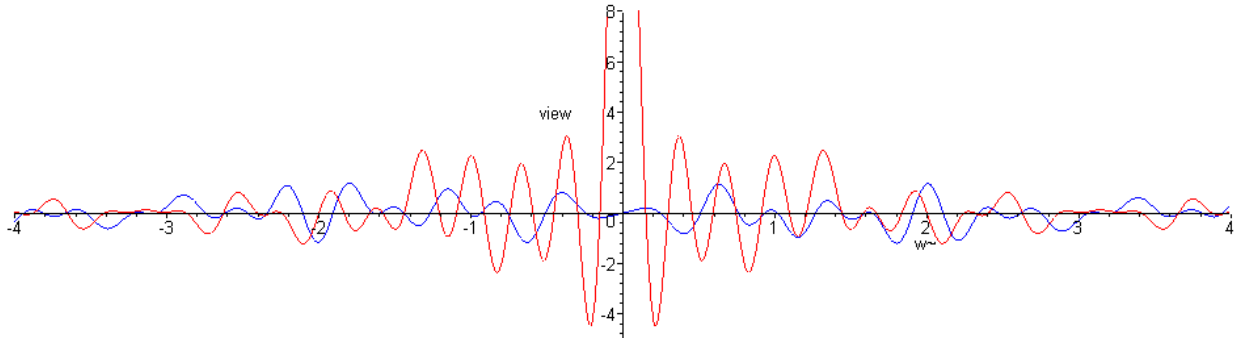



Fig 35.9

**(ℓ) What role has the Autocorrelation Function played in our development?**

In **Section 32** the autocorrelation function was first introduced as $r_x(t)$ for x(t), and was computed for the simple case of a box pulse. Treating the definition of $r_x(t)$ as a convolution equation, the Wiener-Khintchine relation $R_x(\omega) = |X(\omega)|^2$ was trivially derived, where $R_x(\omega)$ is the Fourier Integral Transform of $r_x(t)$. It was then shown that the spectral energy density of a pulse train is $\mathcal{E}(\omega) = (1/2\pi)\, R_x(\omega)$ due to this relation. It was then shown that $r_x(0) = E$, the total energy in the pulse train. We then commented on the origin of the name, showing that the auto-correlation function is the cross-correlation function of a function with itself. Finally, we developed the Z Transform version of the Wiener-Khintchine Relation $R''(z) = (T_1/T)\, |\, Y''(z)\, |^2$ where R"(z) is the Z Transform of the autocorrelation sequence $r_s$.

**Section 33** (simple pulse trains) made no mention of autocorrelation.

In **Section 34** (general pulse trains) it was noted again that $P = r_x(0)/T$ since P = E/T, and that $\mathcal{P}(\omega) = R_x(\omega)/(2\pi T)$ since $R_x(\omega) = |X(\omega)|^2$. At the end of section (b) both $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$ are stated in terms of R"(z) . In particular, (34.14a) says $\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$ where $R''(z) = \Sigma_s\, r_s z^{-s}$, and this expression for $\mathcal{P}(\omega)$ appears throughout the current Section 35.

Comments:

(1) In the Two Pathways discussion above, we showed that the autocorrelation pathway is only one of two approaches one might take to obtain $\mathscr{P}(\omega)$. Thus, it is always possible to compute $\mathscr{P}(\omega)$ without ever using or even knowing about the autocorrelation sequence $r_s$. For example, in (35.17) the autocorrelation approach leads to a nearly intractable sum evaluation (Appendix D), whereas the double-sum approach leads to an easier discovery of $\mathscr{P}(\omega)$, as shown directly in (35.19).

(2) In some textbooks, one gets the impression that the autocorrelation function is somehow indispensable for the development of the $\mathscr{P}(\omega)$ equations. It is not, but it is convenient for many applications. We will use the autocorrelation approach several times in the remaining Sections.