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(f) Power Spectral Density Calculation using the Ensemble Method

The starting point is (35.5) which we repeat here,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \quad . \qquad (35.5)$$

In section (c) we determined these ensemble averages for random variables Y_n and Y_m

$$\langle y_m y_n \rangle = (-p^2/a)\, a^{|m-n|} \qquad m \neq n \qquad // \; a \equiv (1-2p)$$
$$\langle y_n^2 \rangle = p \quad . \qquad (37.8)$$

which we write now as

$$\langle y_m y_{m+s} \rangle = (-p^2/a)\, a^{|s|} \qquad s \neq 0 \qquad // \; a \equiv (1-2p)$$
$$\langle y_n^2 \rangle = p \quad . \qquad (37.8)$$

Then (35.5) may be written as

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \sum_{m=-\infty}^{\infty} \{ p + (-p^2/a) \sum_{s \neq 0} a^{|s|} z^{-s} \}$$

But we calculated {.....} in (37.10) so we don't repeat all that work and just quote the result,

$$\{ p + (-p^2/a) \sum_{s \neq 0} a^{|s|} z^{-s} \} = \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \qquad z = e^{i\omega T_1}$$

Thus we have

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \sum_{m=-\infty}^{\infty} \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \left(\frac{T_1}{T} \sum_{m=-\infty}^{\infty} [1] \right)$$

The remaining sum is either $2\pi\delta(0)$ or $(2N+1)$ in the limit $N \to \infty$. But (T/T_1) is this same quantity, so we find that ( ) = 1 and our final result is

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega) \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)}$$

This is then the mean of the spectral power densities of all the pulse trains in the ensemble. Since we are in the $N \to \infty$ limit, this is the same as $\mathcal{P}(\omega)$ for any pulse train in the ensemble so we end up with

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega) \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \tag{37.15}$$

which is the same as (37.14) as computed by the autocorrelation method. As usual, in the "ensemble method", there is no need to talk about $\langle \ldots \rangle_1$ type averages, or autocorrelation sequences.