## 35. Statistical Pulse Trains

### What is a Statistical Pulse Train?

Suppose we intercept a 1 GHz digital signal (zeros and ones) for 1 μsec and thus grab a sequence of N = 1,000 amplitude symbols $y_m$ (perhaps using a logic analyzer or digital scope). We could compute the average value for this sequence and we might find $\langle y_m \rangle_1 = 0.55$. Here $\langle ... \rangle_1$ refers to a "horizontal" average over a single pulse train amplitude sequence. By its definition, this average cannot depend on the index m, we are just doing an average of the $y_m$ values in the sequence,

$$\langle y_m \rangle_1 = (1/N) \Sigma_{m=1}^{N} y_m \ .$$

Now it might be that during this 1 μsec, there was something unusual about the data being sent, so that this measurement of $\langle y_m \rangle_1 = 0.55$ is not really representative over a longer term. One alternative would be to capture a much longer sequence. Rather than do this, our approach will be to grab very many 1 μsec pulse trains and form an **ensemble** of these representative pulse trains. Perhaps we do this for a week, so the ensemble then has a huge number of pulse trains. We then write down these amplitude sequences in a vertical list on a very tall piece of paper, one sequence below the next. We then number the positions in the sequences 1 to N and we then define $\langle y_m \rangle$ as the *vertical* average through this ensemble of the sequences in position m. This vertical average is written

$$\langle y_m \rangle = (1/I) \Sigma_{i=1}^{I} y_m^{(i)}$$

and in theory this could be different for different columns m. For example, consider this ensemble of seqeunces which has N = 6 and I = 3:

```
1   2   3   4   5   6
a   b   c   d   e   f          sequence #1
a'  b'  c'  d'  e'  f'         sequence #2
a"  b"  c"  d"  d"  f"         sequence #3
```

We would have

$\langle y_m \rangle_1 = (a+b+c+d+e+f)/6.$ // for the first sequence
$\langle y_3 \rangle = (c + c' + c")/3$ // for column 3

Our pulse train sequence has a random variable $Y_m$ associated with each horizontal pulse train position, m = 1,2...N, and the "vertical" average $\langle y_m \rangle$ is the expected value of $Y_m$ over the ensemble, normally written $\langle y_m \rangle = E(Y_m)$. The experiment associated with $Y_m$ is the generation of a pulse train by some Apparatus, the experiment is run many times, and the outcomes for random variable $Y_m$ are the values of the symbols located in position m of these pulse trains. See Appendix G for the meaning of "random variable".

The ensemble is characterized by various statistical properties, such as $E(Y_m)$ or $E(Y_n Y_m Y_k)$, each being a vertical average down through the ensemble. Our **statistical pulse train** is an idealized pulse train whose amplitudes form a **statistical sequence** whose statistics ( like $E(Y_m)$ and $E(Y_n Y_m Y_k)$ ) exactly match those of the ensemble. It is not any particular member of the ensemble, since any member might

deviate somehow. And it is certainly not the average of the pulse trains in the ensemble, since this average pulse train would have amplitudes $y_m = \langle y_m \rangle$ which has nothing to do with anything ( and also would have illegal symbols, eg, $y_m = 1/2$). If N were very large, a pulse train of the ensemble might come close to being a statistical pulse train.

More normal parlance would refer to our statistical pulse train as a **random pulse train** whose sequence of amplitudes is a **random sequence**. The problem with this terminology is that the word "random" suggests for example that $\langle y_m \rangle$ = the mean value of $Y_m$ in the ensemble = 1/2. That is to say, the word random suggests a flat distribution where p(1) = 1/2 and p(0) = 1/2. But if p(1) = 1/3 and p(0) = 2/3, our ensemble would still be associated with a statistical pulse train, Any value of p ≡ p(1) would describe a statistical pulse train, along with the other statistical properties. This same subtle implication of the word random appears in Appendix G on "random variables" and basic probability theory.

In what follows, we shall only be concerned with the first and second order statistics of the statistical pulse train, which are $\langle y_m \rangle = E(Y_m)$ and $\langle y_m y_n \rangle = E(Y_m Y_n)$.

**(a) Spectral power density for a Statistical Pulse Train**

With the above introduction, we imagine now a large ensemble of I pulse trains in which a particular pulse train is labeled by index i. This pulse train has coefficients $y_n^{(i)}$ . We are interested in the ensemble averages of $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$ and P. We indicate the ensemble average of some quantity Q by <Q>. By averaging equations (34.13) through (34.15) we obtain

$$\langle \mathcal{E}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \tag{35.1}$$

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \tag{35.2}$$

$$\langle P \rangle = \int_{-\infty}^{\infty} d\omega \, \langle \mathcal{P}(\omega) \rangle \tag{35.3}$$

where

$$\langle y_m^* \, y_n \rangle = (1/I) \sum_{i=1}^{I} y_m^{(i)*} \, y_n^{(i)} \quad . \tag{35.4}$$

Up to this point we have been general, allowing the $y_m^{(i)}$ to be complex coefficients, but from now on we consider them to be real, so we can delete the asterisks in the above equations.

In general one can consider statistical pulse trains whose amplitudes are randomly ("statistically!") selected in some manner from a discrete set of possible amplitudes {A,B,C....}, or perhaps even from a continuous set. Here we limit our interest to the discrete set {A,B}, but the methods outlined below can be generalized for signals having more than two encoded amplitudes.

A very simple case to consider is this, where A and B are real,

$y_m = A$ probability p p(A)
$y_m = B$ probability 1-p p(B) . (35.5)

Remember that index m labels pulse location m in the pulse train. *If* probabilities of a pulse value at locations m and at n are **independent** (so they are uncorrelated), then we can write the averaged coefficients as follows, first for two different locations $m \neq n$, then for the same location $m = n$ :

$n \neq m$ $\alpha \equiv \langle y_m y_n \rangle = [pp]\, AA + [p(1-p)]\, AB + [(1-p)p]BA + [(1-p)(1-p)]BB$

$n = m$ $\beta \equiv \langle y_m^2 \rangle = [p]AA + [(1-p)]\, BB$ . (35.6)

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. The reader must now stop reading and stare at the above equations until they make complete sense. Notice that the second line is quite distinct from the first line. In the double summation in (35.2), there are both diagonal terms where m = n, and off-diagonal terms where m≠n. These groupings must be treated separately according to (35.6). We have defined new symbols α and β to emphasize that, for the situations shown, there is no dependence on the n and m indices for these quantities. In a later sections, we shall encounter situations where the above equations do not apply because there *is* correlation between positions m and n.

Comment: The non-dependence of $\langle y_m \rangle$ and $\langle y_m^2 \rangle$ on m can be justified loosely by saying that nothing distinguishes one pulse train position from another. This issue is addressed in Appendix G (g), see for example (G.46), (G.47) and (G.50). We find for example that $\langle y_m \rangle$ is position-independent for a suitable ensemble regardless of the nature of the correlation between the random variables for the pulse train.

Inserting (35.6) into (35.2) gives a fundamental result [ with T as in (34.4) ]

$$\langle \mathscr{P}(\omega) \rangle = T_1\, \mathscr{P}_{pulse}(\omega)\, (1/T)\, \{ \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} . \quad (35.7)$$

Appendix D shows how α and β may be written in terms of the mean μ and variance $\sigma^2$ of the values of the amplitudes $y_m^{(i)}$ for the pulses at position m in the pulse train ensemble. There we show that

$\langle y_m \rangle = \mu = [p]A + [1-p]B$

$n \neq m$ $\alpha \equiv \langle y_m y_n \rangle = \langle y_m \rangle^2 = \mu^2 = \{ [p]A + [1-p]B \}^2 = [pp]\, AA + [p(1-p)]\, 2AB + [(1-p)(1-)]BB$

$n = m$ $\beta \equiv \langle y_m^2 \rangle = [p]AA + [(1-p)]\, BB = \sigma^2 + \mu^2 = \sigma^2 + \alpha$ *(D.16)*

$\sigma^2 = [\, p(1-p)]\, (A-B)^2$ . *(D.18)*

The key relations of interest are these,

$\alpha = \mu^2$

$$\beta = \mu^2 + \sigma^2 \quad = \alpha + \sigma^2$$
$$\sigma^2 = (\beta - \alpha) \,. \qquad \textit{(D.17)}$$

Thus we can write (35.7) as

$$\langle \mathscr{P}(\omega) \rangle \;= T_1\, \mathscr{P}_{\mathbf{pulse}}(\omega)\, (1/T)\, \{\; \mu^2 \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \;+ (\mu^2 + \sigma^2) \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \;\} \qquad (35.7a)$$

In general we shall continue to use the parameters α and β, occasionally referring to μ and σ.

**(b) Infinite Statistical Pulse Train with Non-Correlated Coefficients**

Recall now equation (35.7),

$$\langle \mathscr{P}(\omega) \rangle \;= T_1\, \mathscr{P}_{\mathbf{pulse}}(\omega)\, (1/T)\, \{\; \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \;+ \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \;\} \quad . \qquad \textit{(35.7)}$$

To evaluate the first double sum, we write it as

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \;= \sum_{n=-\infty}^{\infty} \{\; \sum_{m=-\infty}^{\infty} [\, e^{i\omega(m-n)T_1}] \;- 1 \;\}$$

$$= \Big( \sum_{n=-\infty}^{\infty} e^{+i\omega n T_1} \Big) \Big( \sum_{m=-\infty}^{\infty} e^{-i\omega m T_1} \Big) - \sum_{n=-\infty}^{\infty} 1 \quad = \Big| \sum_{n=-\infty}^{\infty} e^{+i\omega n T_1} \Big|^2 \;- \sum_{n=-\infty}^{\infty} 1 \quad . \qquad (35.8)$$

Note that each sum is real, according to (13.2), so we can replace $|\;\;|^2$ by $\{\;\}^2$.

We now quote two results from Appendix A,

$$\sum_{n=-\infty}^{\infty} 1 \;= [\, 2\pi\, \delta(0)] \qquad \textit{(A.36)}$$

$$\{ \sum_{n=-\infty}^{\infty} e^{in\omega T_1} \}^2 \;= [\, 2\pi\delta(0)\,] \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \quad , \qquad \textit{(A.39)}$$

so the first double sum in (35.7) becomes

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \;= [\, 2\pi\delta(0)\,] \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \;- 1\} \quad . \qquad (35.9)$$

From (A.36) the second double sum in (35.7) is just $[\, \Sigma_{m=n}\, 1$ for fixed n has just one term which is 1 ]

$$\sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \;= \sum_{n=-\infty}^{\infty} [1] \;= [\, 2\pi\, \delta(0)] \;, \qquad (35.10)$$

and therefore we may write (35.7) as

$$<\mathscr{P}(\omega)> = T_1\, \mathscr{P}_{pulse}(\omega)\,(1/T)\quad \{\quad \alpha\,[\,2\pi\delta(0)\,]\,[\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\;-1] + \beta\,[\,2\pi\,\delta(0)]\quad \}$$

$$= \mathscr{P}_{pulse}(\omega)\,(T_1\,[\,2\pi\delta(0)\,]/T)\,\{\,\alpha\,[\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\;-1] + \beta\,\}$$

or

$$<\mathscr{P}(\omega)> = \mathscr{P}_{pulse}(\omega)\,\{\,(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\} \tag{35.11}$$

where we used $T = [2\pi\delta(0)]T_1$ from (33.22). This is a famous result which will be applied below.

In Appendix D (D.17) we show that $(\beta-\alpha) = \sigma^2$ and $\alpha = \mu^2$ where $\mu$ and $\sigma^2$ are the mean and variance of the pulse train amplitudes $y_m^{(i)}$ which occur at position m in the pulse train. Thus (35.11) may be written,

$$<\mathscr{P}(\omega)> = \mathscr{P}_{pulse}(\omega)\,\{\,\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\}\ . \tag{35.11a}$$

In this form we see that the continuous part of the spectrum arises from the variance $\sigma^2$ in the ensemble values of the pulse train amplitudes $y_m^{(i)}$, while the discrete part of the spectrum is present only if the pulse amplitudes have a non-vanishing mean $\mu$ .

Verification

To find external verification for (35.11), we write things in terms of f where $\omega = 2\pi f$,

$$\mathscr{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = \frac{|\mathscr{X}_{pulse}(f)|^2}{2\pi T_1} \qquad \text{// (34.4) and text after (1.4)}$$

$$P(f) = 2\pi\mathscr{P}(\omega) \qquad \text{// (34.4) last item}$$

$$2\pi\,\delta(\omega T_1 - 2\pi m) = 2\pi\,\delta(2\pi f T_1 - 2\pi m) = (1/T_1)\,\delta(\,f - m/T_1)\ .$$

Then (35.11a) becomes

$$\frac{1}{2\pi}<P(f)> = \frac{|\mathscr{X}_{pulse}(f)|^2}{2\pi T_1}\,\{\,\sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(\,f - m/T_1)\,\}$$

or

$$<P(f)> = \frac{|\mathscr{X}_{pulse}(f)|^2}{T_1}\,\{\,\sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(\,f - m/T_1)\,\} \tag{35.11b}$$

This may be compared to result (A.17) in Appendix A of Xiong which we quote,

$$\Psi_s(f) = \frac{|G(f)|^2}{T}\left(\sigma_a^2 + \frac{m_a^2}{T}\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T})\right) \qquad \text{(A.17)}$$

**(c) Finite Statistical Pulse Train with Non-Correlated Coefficients**

Recall again equation (35.7) but assume now a finite pulse train so the sums are different

$$\langle \mathscr{P}(\omega)\rangle = T_1\, \mathscr{P}_{\mathbf{pulse}}(\omega)\,(1/T)\,\{\ \alpha \sum_{n=-N}^{N}\sum_{m\neq n}[\,e^{i\omega(m-n)T_1}] + \beta\sum_{n=-N}^{N}\sum_{m=n}[1]\ \}\ . \qquad (35.12)$$

To evaluate the first double sum, we write it as

$$\sum_{n=-N}^{N}\sum_{m\neq n}[\,e^{i\omega(m-n)T_1}] = \sum_{n=-N}^{N}\{\ \sum_{m=-N}^{N}[\,e^{i\omega(m-n)T_1}] - 1\ \}$$

$$= (\ \sum_{n=-N}^{N} e^{+i\omega nT_1}\ )(\ \sum_{m=-N}^{N} e^{-i\omega mT_1}\ ) - \sum_{n=-N}^{N} 1 = |\ \sum_{n=-N}^{N} e^{+i\omega nT_1}\ |^2 - (2N+1)\ . \qquad (35.13)$$

where $|\ |^2 = \{\ \}^2$ since (13.3) says each sum is real. From Appendix A we have

$$\sum_{n=-N}^{N} e^{ink} = 2\pi\ \{\ \frac{\sin[(N+1/2)k]}{2\pi\sin(k/2)}\ \} = 2\pi\delta_5(k,N)\ , \qquad \textit{(A.30)}$$

so using (35.13) the first double sum in (35.12) becomes

$$\sum_{n=-N}^{N}\sum_{m\neq n}[\,e^{i\omega(m-n)T_1}] = [2\pi\delta_5(\omega T_1,N)]^2 - (2N+1)\ . \qquad (35.14)$$

The second double sum in (35.12) is just

$$\sum_{n=-N}^{N}\sum_{m=n}[1] = \sum_{n=-N}^{N}[1] = (2N+1) \qquad (35.15)$$

and therefore we may write (35.12) as

$$\langle \mathscr{P}(\omega)\rangle = T_1\, \mathscr{P}_{\mathbf{pulse}}(\omega)\,(1/T)\,\{\ \alpha\,\{\,[2\pi\delta_5(\omega T_1,N)]^2 - (2N+1)\} + \beta(2N+1)\ \}$$

$$= \mathscr{P}_{\mathbf{pulse}}(\omega)\,((2N+1)\,T_1/T)\,\{\ \alpha\,\{\frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} - 1)\} + \beta\ \}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \{ \ (\beta-\alpha) \ + \alpha \ \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \ \}$$

where this time we used $T = (2N+1)T_1$ from (33.22). The factor multiplying α can be replaced using (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \ \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \ = \ \frac{1}{2N+1} \ \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \qquad \textit{(A.20)}$$

to give

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega) \{ \ (\beta-\alpha) \ + \ \alpha \, 2\pi \, \delta_6(\omega T_1,N) \ \} \qquad (35.16)$$

which is the finite pulse train version of (35.11). In the limit $N \rightarrow \infty$ we know from (A.21)

$$\lim_{N\rightarrow\infty} \delta_6(k,N) \ = \ \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad \textit{(A.21)}$$

so that (35.16) becomes

$$<\mathcal{P}(\omega)> \ = \mathcal{P}_{\mathbf{pulse}}(\omega) \{ (\beta-\alpha) \ + \ \alpha \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1-2\pi m) \}$$

which reproduces (35.11).

**(d) Statistical Non-Correlated Pulse Trains: Summary and Examples**

Here then is a brief summary of the above results:

**Spectral Power Density of a Non-Correlated Statistical Pulse Train** (35.17)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n \, x_{\mathbf{pulse}}(t - nT_1) \qquad \text{// or } \sum_{n=-N}^{N} \text{ for a finite pulse train}$$

$$<\mathcal{P}(\omega)> \ = \mathcal{P}_{\mathbf{pulse}}(\omega) \, [ \, (\beta-\alpha) \ + \ \alpha \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) \, ] \qquad \text{// infinite} \qquad \textit{(35.11)}$$

$$<\mathcal{P}(\omega)> \ = \mathcal{P}_{\mathbf{pulse}}(\omega) \, [ \, (\beta-\alpha) \ + \ \alpha \, 2\pi \, \delta_6(\omega T_1,N) \, ] \qquad \text{// finite} \qquad \textit{(35.16)}$$

$$\alpha = \ [pp] \, AA + [p(1-p)] \, AB + [(1-p)p]BA + [(1-p)(1-p)]BB \qquad \text{// } = <y_m y_n> \text{ when } n \neq m \qquad \textit{(35.6)}$$

$$\beta = \ [p]AA + [(1-p)] \, BB \qquad \text{// } = <y_m y_n> \text{ when } n = m$$

A pulse has probably p of having amplitude A, and probability 1-p of having amplitude B.

A=1 and B=0 $\Rightarrow$ $\alpha = p^2$ $\quad \beta = p$ $\quad (\beta - \alpha) = p(1-p)$

One can express $\alpha$ and $\beta$ in terms of the statistical mean $\mu$ and variance $\sigma^2$ :

$$\mu = [p]A + [1-p]B \qquad \alpha = \mu^2$$
$$\sigma^2 = [\,p(1-p)]\,(A-B)^2 \qquad \beta = \mu^2+\sigma^2 \qquad (\beta-\alpha) = \sigma^2 \qquad (D.16)$$

Example 1: Assuming A = 1 and B = 0, setting p = 1 in the above box gives $\alpha = 1$ and $(\beta-\alpha) = 0$. In this limit, our statistical average pulse train spectrum becomes the same as that for the simple pulse train of (33.25)

$$\mathscr{P}(\omega) \equiv \mathscr{P}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules}$$
$$\mathscr{P}(\omega) \equiv \mathscr{P}_{\mathbf{pulse}}(\omega)\; 2\pi\,\delta_6(\omega T_1, N) \qquad \text{joules} \qquad (33.25)$$

This is because with p = 1 every pulse in the pulse train has the same amplitude A=1, which is how a simple pulse train is defined. As we reduce p below p = 1, the line spectra are scaled down by $\alpha = p^2 < 1$, and a continuous spectrum starts to appear with $(\beta - \alpha) = p(1-p)$. The randomness (variance $\sigma^2$) of the ensemble creates a continuous component in the spectral power density $\mathscr{P}(\omega)$ as per (35.11a) .

Example 2: Let p=0 with A=1 and B=0. Then every pulse has B = 0, $x(t) \equiv 0$, $\alpha = \beta = 0$, and both terms in $\mathscr{P}(\omega)$ vanish so there is zero spectral energy density.

Example 3: Let p=1/2 with A=1 and B=0, so $\alpha = p^2 = 1/4$ and $\beta = p = 1/2$. This pulse train then has an equal probably of 1's and 0's. We find from box (35.17) that

$$\langle\mathscr{P}(\omega)\rangle = \mathscr{P}_{\mathbf{pulse}}(\omega)\,[(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,] \qquad \text{infinite} \qquad (35.18)$$
$$\langle\mathscr{P}(\omega)\rangle = \mathscr{P}_{\mathbf{pulse}}(\omega)\,[(1/4) + (1/4)\,2\pi\,\delta_6(\omega T_1, N)\,] \qquad \text{finite} \qquad (35.19)$$

In (35.18) the discrete spectrum has been reduced to 1/4 of its full strength, and a continuous spectrum exists with coefficient 1/4 as shown.

**(e) A numerical example of a Statistical Pulse Train**

Recall from box (34.4) that for a finite pulse train,

$$\mathscr{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T} \qquad \mathscr{P}_{\mathbf{pulse}}(\omega) = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \qquad T = (2N+1)T_1. \qquad (35.20)$$

Therefore we can write our p = 1/2 Example 3 result (35.19) as

$$\frac{\langle|X(\omega)|^2\rangle}{(2N+1)} = |X_{\mathtt{pulse}}(\omega)|^2\,[(1/4) + (1/4)\,2\pi\,\delta_6(\omega T_1,N)\,] \quad . \tag{35.21}$$

We shall use for $x_{\mathtt{pulse}}(t)$ a square pulse of height A=1 and width $\tau = T_1 = 1$ so that, from (9.2),

$$|X_{\mathtt{pulse}}(\omega)| = \mathrm{sinc}(\omega/2) \quad . \tag{35.22}$$

Since our pulse train will be fairly short (N = 20 pulses) and since we shall only average a small number of pulse trains (M = 10), we know our result will not exactly match (35.21). Still, we hope to see in our result some kind of continuous background spectrum which approximates the curve $(1/4)|X_{\mathtt{pulse}}(\omega)|^2 = (1/4)\,\mathrm{sinc}^2(\omega/2)$, and we expect to see delta-function-like peaks which, since $2\pi\delta_6(0,N) = (2N+1)$, have a peak value of about $(1/4)41 = 10.25$. Since this will be added to the continuous background, the peak should have a height of $10.25 + .25 = 10.5$. However, for our small ensemble, we won't have exactly p = 1/2, so the delta peak won't be exactly 10.5 units high.

We know that $\delta_6$ has identical peaks spaced by $2\pi$, but we expect the non-central peaks to be suppressed by the $\mathrm{sinc}^2(\omega/2)$ zeros which occur at $\omega = n(2\pi)$.

Here is the self-documented Maple program which generates $\langle|X(\omega)|^2\rangle$ . The program also generates the quantity $\langle X(\omega)\rangle$ upon which we shall comment in Section 36 below.

```
restart; assume(w,real); with(plots):
T1 is pulse duration, N is number of pulses in each pulse train
T1 := 1: N := 20:
Spectrum for standard box shaped pulse
Xpulse := T1*sin(w*T1/2)/(w*T1/2):
Subsequent calls to r() will generate either 0 or 1 with prob p = 1/2
r := rand(0..1):
Clear averaging accumulators. M is number of pulse trains to average
Xas means X absolute-value squared
X_acc := 0:Xas_acc := 0: M := 10:
for m from 1 to M do
   Create a random pulse train
   for n from -N to N do y[n] := r(); od:
   Compute the Digital Fourier Transform of the pulse train from (23.5)
   Yp := T1*sum(y[j]*exp(-I*w*j*T1),j=-N..N):
   Compute X(w) for the pulse train from (25.4), omitting Xpulse(w)
   X := (1/T1)* Yp;
   Accumulate both X(w) and |X(w)|^2, divide by M later for ensemble average
   X_acc := X_acc + X;
   Xas_acc := Xas_acc + eval(abs(X)^2);
od:
Compute <X(w)> and <|X(w)|^2> from the accumulators, but Xpulse not yet added
X_av_    :=  X_acc/M:
Xas_av_  := Xas_acc/M:
```

At this point, before $X_{pulse}(\omega)$ is added to the result, we plot $\langle|X(\omega)|^2\rangle$. As expected, we see the peaks of $\delta_6$ spaced by $2\pi$ and having height around 10 units,

**Plot <|X(w)|^2>/(2N+1) without the pulse shape**

```
plot(Xas_av_/(2*N+1), w=-10..10,color = [red,blue],view = 0..12,numpoints = 500);
```

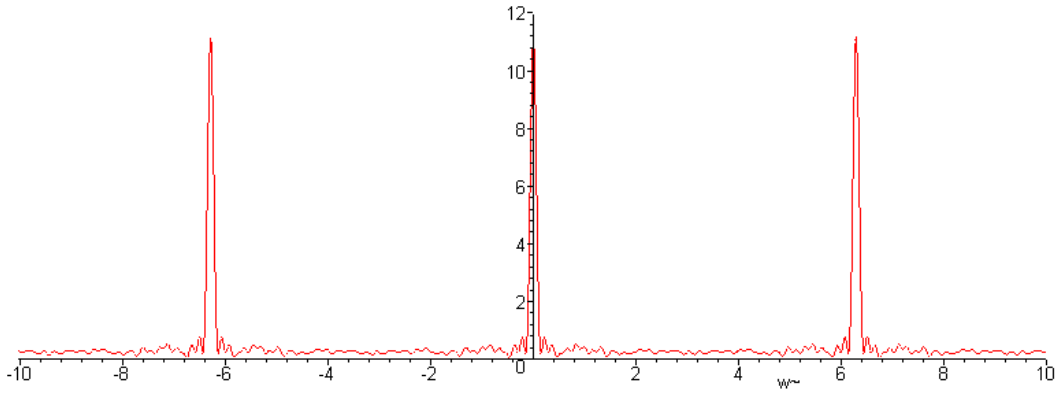



Fig 35.1

We now insert copies of $X_{pulse}(\omega)$ as appropriate,

**Install the pulse shape to get the true expressions for <X(w)> and of <|X(w)|^2>**

```
X_av := Xpulse*X_av_:
Xas_av := Xpulse^2*Xas_av_:
```

and then we can plot $\frac{\langle|X(\omega)|^2\rangle}{(2N+1)}$ for $\omega$ in the same range (-10,10)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..12 ,color = [red,blue],numpoints = 500);
```

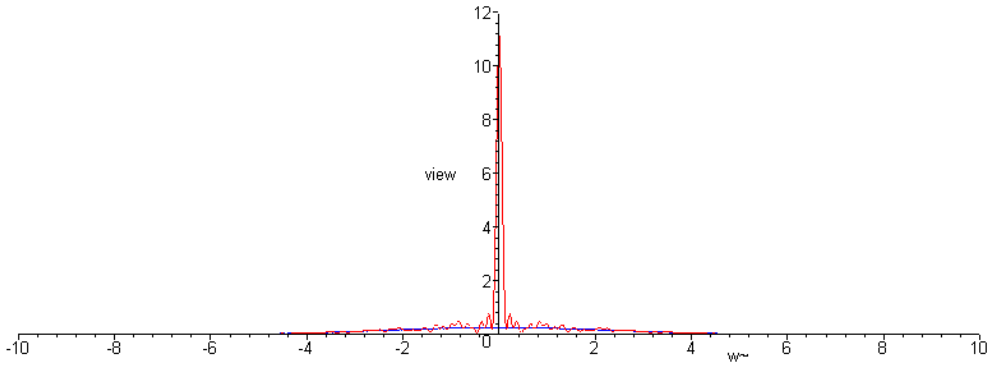



Fig 35.2

We see that the zeros of the sinc function have killed off the adjacent peaks.

Next, we restrict the plot height to be 0.8 units to view the detail, chopping off the $\delta_6$ peak,

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..0.8 ,color = [red,blue],numpoints = 500);
```

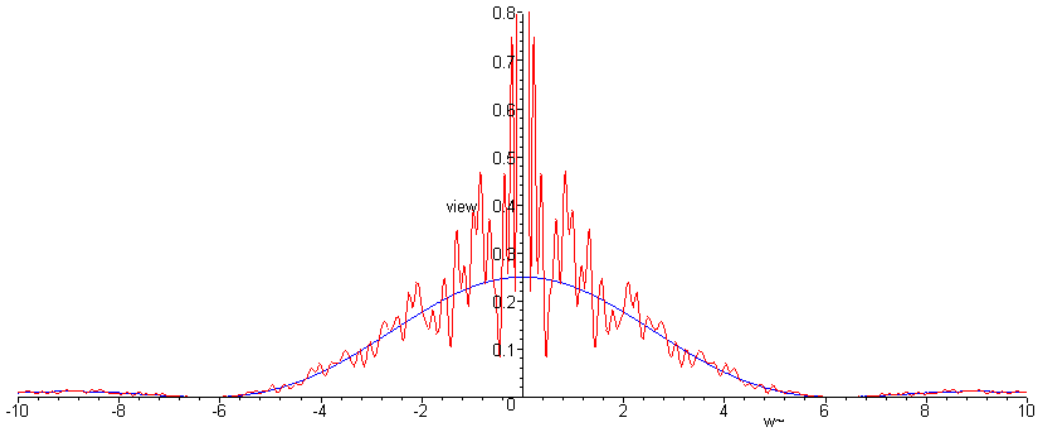



Fig 35.3

The spectrum $\frac{<|X(\omega)|^2>}{(2N+1)}$ is seen to have a continuous component which very well approximates one quarter of the sinc$^2$ curve, as we hoped it would. This tracking also occurs away from the central peak. Here is a blow-up of the above plot for ω in the range (5, 30)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=5..30, view=0..0.02 ,color = [red,blue],numpoints = 500);
```

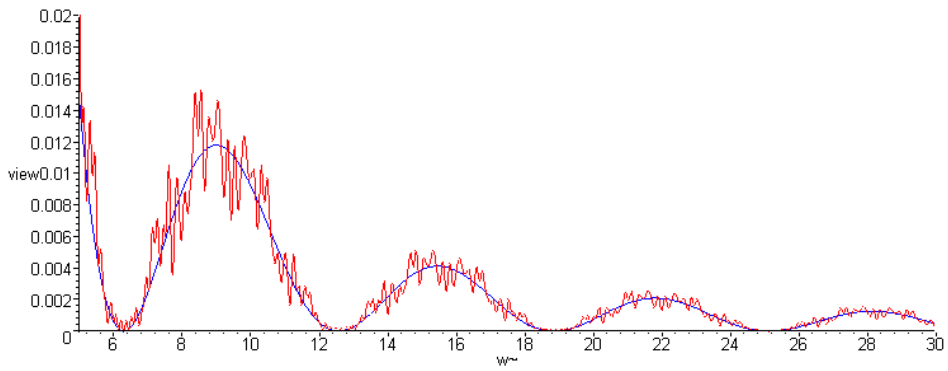



Fig 35.4

It might be noted that Maple does this work analytically, so that Was-av is a function of ω having a large number of trigonometric terms. For the reader's interest, we show Was-av(ω) for a typical program run :

$$Was_av := \frac{2}{5}\sin\left(\frac{1}{2}w\right)^2 ((2\cos(8w) + 2\cos(10w) + \cos(9w) + \cos(12w) + 2\cos(11w) + \cos(14w) + \cos(17w) + 2\cos(5w) + \cos(4w) + 2\cos(3w)$$
$$+ \cos(w) + \cos(2w) + 2\cos(20w) + 2\cos(18w) + \cos(19w) + \cos(15w) + \cos(13w))^2$$
$$+ (\sin(9w) + \sin(12w) + \sin(14w) + \sin(17w) - \sin(4w) + \sin(w) + \sin(2w) - \sin(19w) - \sin(15w) - \sin(13w))^2 + (1 + 2\cos(8w) + \cos(10w)$$
$$+ 2\cos(11w) + 2\cos(13w) + 2\cos(14w) + 2\cos(16w) + \cos(6w) + 2\cos(5w) + 2\cos(4w) + \cos(2w) + \cos(20w) + 2\cos(19w) + \cos(17w) + \cos(9w)$$
$$+ \cos(7w))^2 + (\sin(10w) - \sin(6w) - \sin(2w) - \sin(20w) - \sin(17w) - \sin(9w) - \sin(7w))^2 + (1 + 2\cos(7w) + \cos(9w) + \cos(13w) + \cos(14w)$$
$$+ 2\cos(15w) + \cos(8w) + 2\cos(6w) + 2\cos(5w) + \cos(4w) + 2\cos(3w) + 2\cos(2w) + 2\cos(w) + \cos(20w) + \cos(18w) + \cos(12w) + \cos(19w))^2$$
$$+ (\sin(9w) + \sin(13w) + \sin(14w) - \sin(8w) - \sin(4w) - \sin(20w) - \sin(18w) - \sin(12w) + \sin(19w))^2 + (1 + 2\cos(10w) + \cos(13w) + 2\cos(14w)$$
$$+ 2\cos(15w) + 2\cos(16w) + \cos(17w) + \cos(8w) + 2\cos(5w) + \cos(2w) + 2\cos(w) + \cos(6w) + 2\cos(18w) + \cos(19w) + \cos(9w) + \cos(7w)$$
$$+ \cos(20w))^2 + (\sin(13w) + \sin(17w) - \sin(8w) - \sin(2w) + \sin(6w) - \sin(19w) - \sin(9w) - \sin(7w) + \sin(20w))^2 + (\cos(7w) + \cos(10w) + \cos(9w)$$
$$+ 2\cos(12w) + 2\cos(13w) + \cos(15w) + \cos(5w) + 2\cos(3w) + \cos(w) + \cos(4w) + \cos(6w) + 2\cos(19w) + \cos(11w) + \cos(20w))^2$$
$$+ (\sin(7w) + \sin(10w) + \sin(9w) + \sin(15w) - \sin(5w) - \sin(w) + \sin(4w) + \sin(6w) - \sin(11w) + \sin(20w))^2 + (1 + 2\cos(7w) + \cos(12w) + \cos(13w)$$
$$+ \cos(14w) + 2\cos(15w) + \cos(16w) + \cos(8w) + 2\cos(5w) + 2\cos(3w) + 2\cos(w) + \cos(2w) + \cos(4w) + \cos(20w) + \cos(17w) + \cos(9w)$$
$$+ \cos(18w))^2 + (\sin(12w) + \sin(13w) + \sin(14w) + \sin(16w) - \sin(8w) + \sin(2w) + \sin(4w) - \sin(20w) - \sin(17w) - \sin(9w) + \sin(18w))^2 + (1$$
$$+ 2\cos(7w) + \cos(12w) + 2\cos(11w) + \cos(14w) + 2\cos(17w) + \cos(8w) + 2\cos(6w) + \cos(5w) + \cos(4w) + 2\cos(3w) + \cos(2w) + \cos(w)$$
$$+ \cos(20w) + \cos(16w) + \cos(9w))^2 + (\sin(12w) + \sin(14w) - \sin(8w) - \sin(5w) - \sin(4w) - \sin(2w) + \sin(w) - \sin(20w) - \sin(16w) - \sin(9w))^2 + (1$$
$$+ 2\cos(7w) + \cos(8w) + 2\cos(12w) + 2\cos(13w) + \cos(14w) + 2\cos(17w) + 2\cos(6w) + \cos(5w) + \cos(4w) + \cos(3w) + 2\cos(w) + 2\cos(18w)$$
$$+ \cos(11w) + \cos(10w))^2 + (\sin(8w) + \sin(14w) - \sin(5w) - \sin(4w) - \sin(3w) - \sin(11w) - \sin(10w))^2$$
$$+ (\cos(8w) + 2\cos(10w) + \cos(13w) + 2\cos(6w) + \cos(5w) + \cos(w) + \cos(2w) + \cos(3w) + \cos(20w) + \cos(17w) + \cos(12w) + \cos(9w) + \cos(7w))^2$$
$$+ (\sin(8w) + \sin(13w) - \sin(5w) - \sin(w) + \sin(2w) + \sin(3w) - \sin(20w) - \sin(17w) - \sin(12w) - \sin(9w) - \sin(7w))^2 + (\cos(8w) + \cos(10w)$$
$$+ 2\cos(14w) + \cos(16w) + \cos(17w) + \cos(6w) + \cos(5w) + 2\cos(4w) + \cos(3w) + \cos(w) + \cos(20w) + 2\cos(18w) + \cos(19w) + \cos(12w)$$
$$+ \cos(7w))^2 + (\sin(8w) + \sin(10w) + \sin(16w) + \sin(17w) - \sin(6w) - \sin(5w) - \sin(3w) + \sin(w) - \sin(20w) - \sin(19w) - \sin(12w) - \sin(7w))^2) \Big/ w^2$$

In more serious work with larger numbers, one would of course do this in a more numeric fashion, but we are able to confirm the basic results even with this small experiment.

**(f) What role has the Autocorrelation Function played in our development?**

In Section 32 the autocorrelation function was first introduced as $r_{\mathbf{x}}(t)$ for x(t), and was computed for the simple case of a box pulse. Treating the definition of $r_{\mathbf{x}}(t)$ as a convolution equation, the Wiener-Khintchine Relation $R_{\mathbf{x}}(\omega) = |X(\omega)|^2$ was trivially derived. It was then shown that the spectral energy density of a pulse train is $\mathcal{E}(\omega) = (1/2\pi)\, R_{\mathbf{x}}(\omega)$ due to this relation. Finally, it was shown that $r_{\mathbf{x}}(0) = E$, the total energy in the pulse train. We then commented on the origin of the name, showing that the auto-correlation function is the cross-correlation function of a function with itself when that function is real.

In Section 34 it was noted again that $P = r_{\mathbf{x}}(0)/T$ since P = E/T, and that $\mathcal{P}(\omega) = R_{\mathbf{x}}(\omega)/(2\pi T)$ since $R_{\mathbf{x}}(\omega) = |X(\omega)|^2$.

Sections 33 and 35 made no reference at all to the autocorrelation function.

This leads us to make several comments:

(1) The autocorrelation function played no role whatsoever in our development of key equations such as

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\, [\, (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\, ] \qquad \text{// infinite} \qquad (35.11)$$

(2) In some textbooks, one gets the impression that the autocorrelation function is somehow crucial for the development of such equations. It is not.

(3) Nevertheless, since $\mathcal{P}(\omega) = R_x(\omega)/(2\pi T)$, one can start with a description of $x^{(i)}(t)$ of pulse train i in a statistical ensemble, compute from it $r_x^{(i)}(t)$ and from that $R_x^{(i)}(\omega)$ . One could then do a statistical average to obtain $\langle\mathcal{P}(\omega)\rangle = (2\pi T)^{-1}(1/I)\,\Sigma_{i=1}^{I} R_x^{(i)}(\omega)$. So it is possible to take a pathway to deriving equations like (35.11) which does pass through the land of the autocorrelation function.

(4) Our main reason for even bringing it up is that autocorrelation is closely related to what we are doing, and in other applications such as those involving noise (and pseudo-noise PN) it becomes more significant. In such applications, the definition of $r_x(t)$ might include a normalizing factor so it is then autocorrelation per unit time, or per pulse (per chip in the PN world).

(5) In (34.14a) we noted that $\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ R''(z)$ where R"(z) is the Z transform of the autocorrelation *sequence* $r_s$ defined in (32.17). Then in Appendix F (e) we compute R"(z) for a sequence which respects certain simple conditions met by a Maximum Length Sequence (MLS). The resulting $\mathcal{P}(\omega)$ is not easily computed in any other manner, so in this case the pathway through the autocorrelation route is extremely valuable.

**(g) A paradox and its resolution**

Now while here, we can back up and apply our statistical average directly to the spectrum in (34.6) using (34.7). The result is

$$\langle X(\omega)\rangle = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \langle y_n\rangle\ e^{-in\omega T_1} \quad . \qquad (34.6,7)$$

We could then argue that

$$\langle y_n\rangle = [p]\ A + [1-p]\ B = p \qquad \text{if } A=1,\ B=0 \quad . \qquad (35.23)$$

In this case, we can extract p from the above sum, which then collapses to form the usual (13.2) delta function sum. The result is then the same as the regular pulse train result (14.4) with an overall factor of p out front, namely,

$$\langle X(\omega)\rangle = p \sum_{m=-\infty}^{\infty} X_{pulse}(m\omega_1)\ 2\pi\ \delta(\omega T_1 - 2\pi m) \ . \qquad (35.24)$$

This says that our average spectrum is 100% *discrete*, there is no continuous part! If p = 1/2, the average spectrum is just 1/2 times our discrete unit amplitude pulse train spectrum (14.4). How can this be true, if we just showed in (35.11) that the average spectral density $\langle|X(\omega)|^2\rangle$ has a *continuous* spectral component?

The answer lies in the fact that $\langle ab\rangle \neq \langle a\rangle\langle b\rangle$, where $\langle\ \rangle$ is our averaging operation. Thus

$$\langle|X(\omega)|^2\rangle \neq \langle X(\omega)\rangle\langle X(\omega)^*\rangle \ . \qquad (35.25)$$

There is no reason in the world why the average of a product should be the product of the averages,

$$\left(\frac{1}{N}\Sigma_{i=1}^{N} A_i B_i\right) \neq \left(\frac{1}{N}\Sigma_{i=1}^{N} A_i\right)\left(\frac{1}{N}\Sigma_{i=1}^{N} B_i\right) \ .$$

In the numerical example presented in Section 35 (f), we computed <X(ω)> for a small ensemble of pulse trains. Here are plots of the real and imaginary part of <X(ω)>,

```
plot([Re(X_av),Im(X_av)], w=-10..10, view =-6..22 ,color = [red,blue],numpoints = 500);
```

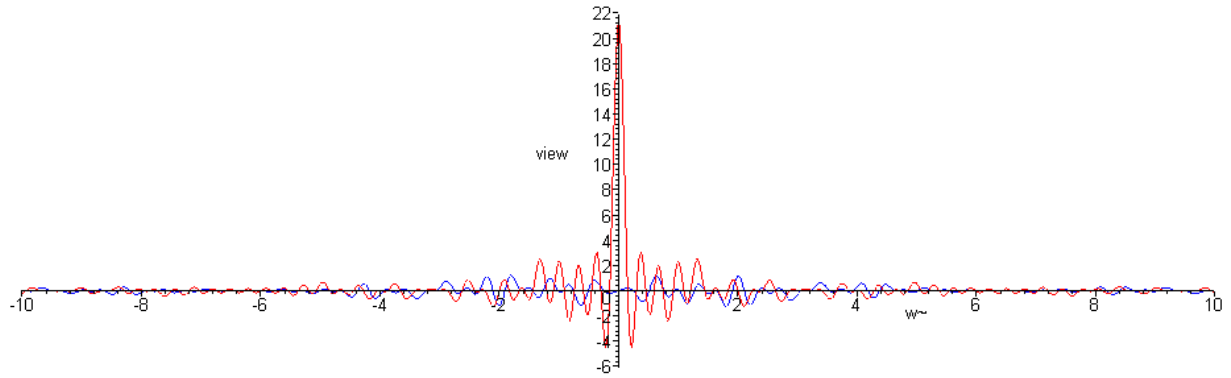



Fig 35.5

We can see that, apart from the noise of our low statistics, there is only the central peak and no continuous spectrum component. A blow-up of the central region follows,

```
plot([Re(X_av),Im(X_av)], w=-4..4, view =-6..8 ,color = [red,blue],numpoints = 500);
```

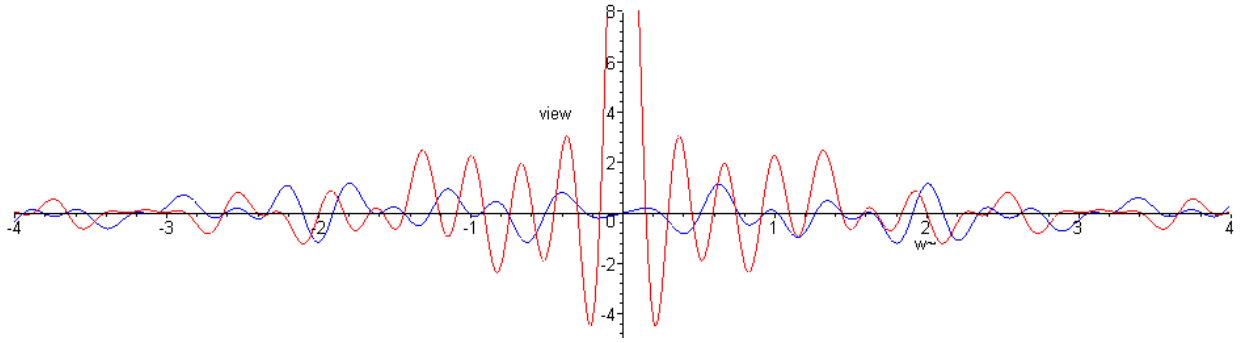



Fig 35.6