37. The AMI Line Code

Pulse Shape. The pulse shape is the same as for unipolar NRZ ,

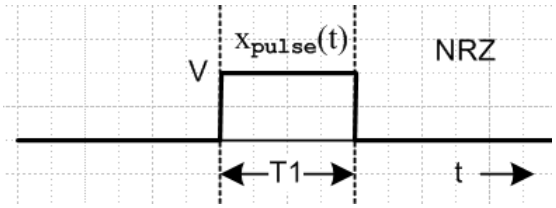


Fig 36.2

$$X_{pulse}(\omega) = (VT_1)\,\mathrm{sinc}(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = (1/2\pi)\,V^2 T_1\,\mathrm{sinc}^2(\omega T_1/2) \qquad (36.1)$$

However, we shall do the analysis below for a general $x_{pulse}(t)$ and insert the box shape at the end.

Coding: Alternate Mark Inversion (AMI) means that a 0 is encoded as a zero (for duration T_1) and a 1 is encoded as a pulse (of duration T_1) of either plus or minus polarity. As each 1 is encountered in the data, the pulse polarity is the negative of that used for the previous encoded 1 pulse, so the 1 polarities are alternated, as in this example

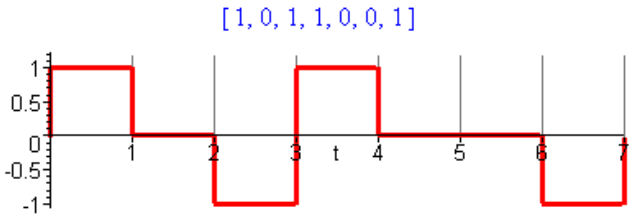


Fig 37.1

Coefficients $\alpha_{m,n}$ and β: 0 is coded with amplitude B = 0, but a 1 is coded with either A = +1 or A = -1, so we have A = ± 1, B = 0. Because there is now correlation between different locations m and n in the pulse train, we can no longer use the simple results of box (35.17). We have to back up to an earlier point in the development. Our starting point will be equation (35.2) which we repeat here, averaging over the statistical ensemble (y_n real),

$$\langle \mathcal{P}(\omega) \rangle = T_1\,\mathcal{P}_{pulse}(\omega)\,(1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_{n,m}\, e^{i\omega(m-n)T_1} \quad . \qquad (35.2)$$

where $\alpha_{n,m} \equiv \langle y_m\, y_n \rangle$ and $\beta \equiv \langle y_n^2 \rangle = \alpha_{n,n}$

Breaking the double sum into two terms as done in Section 35, we obtain this new version of (35.7), valid when there is correlation between pulse train positions m and n,

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \sum_{n=-\infty}^{\infty} \sum_{m \neq n} \alpha_{n,m} [e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} . \qquad (37.1)$$

We develop things as usual for a general pulse shape, but in the end we will use a square pulse as shown in the figure above. The double sum in (35.2) is $\langle | Y''(z) |^2 \rangle$ where Y''(z) is the Z Transform of y_n from box (24.37).

Consider now the expressions given in (35.6) for the statistical averages $\langle y_n^2 \rangle$ and $\langle y_m y_n \rangle$. We assume that p is the probability of a 1 being coded, so 1-p is the probability of a 0 being coded. Then we have

$$\langle y_n^2 \rangle = [p]AA + [(1-p)] \, BB = [p]AA = [p] \, (\pm 1) \, (\pm 1) = p \qquad (37.2)$$

That was the easy one.

For $m \neq n$, we have a *much* harder problem. Consider

$$\langle y_m y_n \rangle = [pp] \, AA' + [p(1-p)] \, AB' + [(1-p)p]BA' + [(1-p)(1-p)]BB'$$

$$= [pp] \, \sigma \, \sigma' + [p(1-p)] \, \sigma \, 0 + [(1-p)p] \, 0 \, \sigma' + [(1-p)(1-p)] \, 0 \, 0$$

$$= p^2 \, \sigma \, \sigma'$$

where $\sigma = \pm 1$ and $\sigma' = \pm 1$. Here p^2 is the probability that both slot positions y_m and y_n are coded for 1. This can happen in four different ways, as illustrated here,

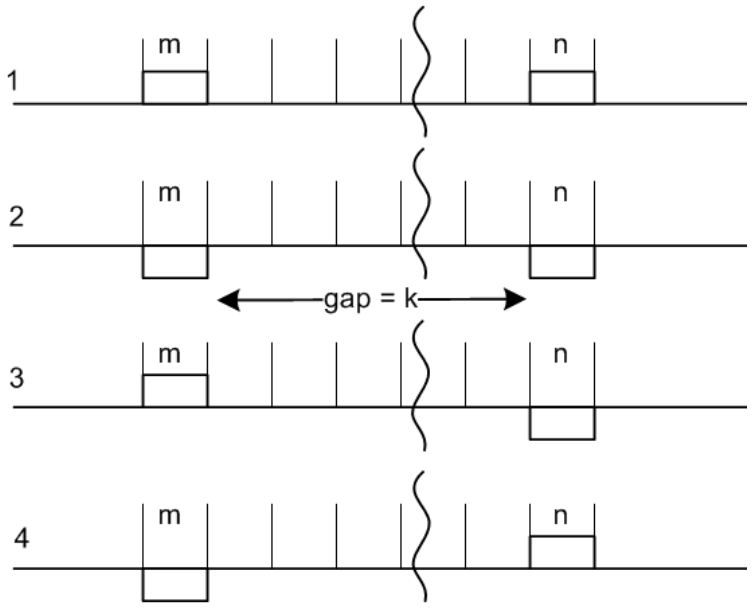


Fig 37.2

By symmetry, the probability of cases 1 and 2 is the same, and the probability of cases 3 and 4 is the same. This is perhaps not totally obvious, but the reason will become clear below when we talk about legal pulse patterns.

Given that both m and n are coded for 1, let X/2 be the total probability for the case 1, and Y/2 be the total for the case 3. Then we can write

$$\langle y_m y_n \rangle = (+1)(+1)\, p^2 X/2 + (-1)(-1)\, p^2 X/2 + (+1)(-1)\, p^2 Y/2 + (-1)(+1)\, p^2 Y/2$$

$$= p^2(X-Y) .$$

Given that both m and n are coded for a 1, since we have enumerated all the cases, we must have

$$X + Y = 1 \qquad \text{// probability of getting any of the four cases.}$$

Our task then is to compute probabilities X and Y.

For cases 1 and 2 taken together, X is the probability that the gap between the coded 1's is filled with a legal sequence of pulses. This is the key statement and the reader may want to ponder the previous sentence thinking about probability as the number of legal ways divided by the total number of ways. Only the legal ways can show up in a statistical ensemble.

If the gap is "legal", there must be an *odd* number of coded 1's in the gap, due to the AMI alternation coding rule. Similarly, Y is the probability that there are an *even* number of coded 1's in the gap. Define,

$$k = |m-n| - 1 = \text{size of gap}$$

and think of X and Y as depending on k, so we write X_k and Y_k.

Note that $Y = Y_k = (1-X_k)$ = probability that gap has *even* number of coded 1's. So far, we add k labels to our results shown above,

$$\langle y_m y_n \rangle = p^2(X_k - Y_k) = p^2 (2X_k - 1) \qquad k = |m-n| - 1 \quad . \tag{37.3}$$

Assume we have a gap of size k and there exists some X_k and Y_k we don't yet know. What can be said about X and Y if the gap is increased to size k+1 by adding one more pulse period in between? Claim:

$$X_{k+1} = Y_k\, p + X_k\, (1-p) \quad = \text{ probability of having an odd number of coded 1's in gap k+1}$$

Explanation:

- Y_k is the probability the k gap had an even number of coded 1's. In order to make the k+1 gap have an odd number of coded 1's we have to put a coded 1 in the new space, which has probability p.
This gives the first term $Y_k\, p$.

• X_k is the probability the k gap had an odd number of coded 1's. In order to make the k+1 gap have an odd number of coded 1's we have to put a coded 0 in the new space, which has probability (p-1).
This gives the second term X_k (1-p) .

Since this exhausts the ways we can get from k to k+1, X_{k+1} has the probability shown above. We could write a similar expression for Y_{k+1} but it is not needed. Since $Y_k = 1-X_k$ we then have

$$X_{k+1} = (1-X_k)\, p + X_k\,(1-p) = p - pX_k + X_k - pX_k = (1-2p)X_k + p \quad . \tag{37.4}$$

Now define,

$$a \equiv (1-2p) \qquad \Rightarrow \; p = (1-a)/2 \quad \text{and } 1-p = (1+a)/2$$

Then the above reads,

$$X_{k+1} = aX_k + p \; . \tag{37.5}$$

This is a difference equation (recurrence relation) which we want to solve for X_k. If there is no gap at all (k=0), we have two adjacent identical pulses which is illegal so $X_0 = 0$. If the gap is k = 1, then the middle element must be different from the two ends, so $X_1 = p$, consistent with (37.5). We now examine the recurrence relations:

$X_0 = 0$
$X_1 = p$
$X_2 = a(p) + p = p(a+1)$
$X_3 = a[p(a+1]+ p = p(a^2+a+1)$
....
$X_k = p\,(a^{k-1} + ... + a^2 + a + 1)$

The geometric series can be summed in the usual manner and yields

$$X_k = p\,(1 - a^k)/(1-a) = p\,(1 - a^k)/2p = (1 - a^k)/2 \; . \tag{37.6}$$

The same result can be obtained from Maple in this manner :

```
p := (1-a)/2:
Xk := rsolve({ X(k+1) = a*X(k) + p,X(0) = 0}, X): simplify(%);
```

$$-\frac{1}{2}a^k + \frac{1}{2}$$

Inserting this result into (37.3) gives

$$\langle y_m y_n \rangle = p^2\,(2X_k - 1) = p^2\,(2[(1 - a^k)/2] - 1) = p^2\,((1 - a^k) - 1) = -p^2 a^k$$

$$= -p^2\, a^{[|m-n| - 1]} = (-p^2/a)\, a^{|m-n|} \quad . \tag{37.7}$$

Therefore, we have our final results for the coefficients

$$\alpha_{m,n} \equiv \langle y_m y_n \rangle = (-p^2/a)\, a^{|m-n|} \qquad m \neq n \qquad\qquad // \; a \equiv (1-2p)$$

$$\beta \equiv \langle y_n^2 \rangle = p \; . \tag{37.8}$$

Spectrum. The starting point is (37.1) which we repeat here,

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega)\, (1/T) \left\{ \sum_{n=-\infty}^{\infty} \sum_{m \neq n} \alpha_{n,m} \left[e^{i\omega(m-n)T_1} \right] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \right\} . \tag{37.1}$$

The second double sum is the same as in the uncorrelated case,

$$\sum_{n=-\infty}^{\infty} \sum_{m=n} [1] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta_{m,n}\, [1] = \sum_{n=-\infty}^{\infty} [1]$$

which we set later to $[2\pi\delta(0)]$ or $(2N+1)$ depending on whether we have an infinite or finite pulse train.

The first double sum is different and we write it using (37.8),

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} \alpha_{n,m} \left[e^{i\omega(m-n)T_1} \right] = \sum_{n=-\infty}^{\infty} \sum_{m \neq n} (-p^2/a)\, a^{|m-n|}\, b^{(m-n)} \qquad\qquad b \equiv e^{i\omega T_1} \; .$$

Ignoring for the moment the $(-p^2/a)$ factor, we painfully process the double sum in many steps,

$$= \sum_{n=-\infty}^{\infty} \sum_{m \neq n} a^{|m-n|}\, b^{(m-n)} \tag{37.9}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{m=-\infty}^{n-1} (b/a)^{m-n} + \sum_{m=n+1}^{\infty} (ab)^{m-n} \right] \qquad s = m-n+1 \qquad r = m-n-1$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{s=-\infty}^{0} (b/a)^{s-1} + \sum_{r=0}^{\infty} (ab)^{r+1} \right] = \sum_{n=-\infty}^{\infty} \left[\sum_{s=0}^{\infty} (b/a)^{-s-1} + \sum_{r=0}^{\infty} (ab)^{r+1} \right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{s=0}^{\infty} (a/b)^{s+1} + \sum_{r=0}^{\infty} (ab)^{r+1} \right] = \left[(a/b) \sum_{s=0}^{\infty} (a/b)^{s} + (ab) \sum_{r=0}^{\infty} (ab)^{r} \right] \sum_{n=-\infty}^{\infty} [1]$$

$$= \left[(a/b)\, \frac{1}{1-a/b} + (ab)\, \frac{1}{1-ab} \right] \sum_{n=-\infty}^{\infty} [1] = \left[\frac{a}{b-a} + \frac{ab}{1-ab} \right] \sum_{n=-\infty}^{\infty} [1] = \frac{a(1+b^2-2ab)}{(b-a)(1-ab)} \sum_{n=-\infty}^{\infty} [1] \, .$$

We note that this double sum is valid for $|a| < 1$ since $|b| = 1$. In the complex a-plane, the circle of convergence is $|a| = 1$. The sum is valid in the limit $a \to -1$, but as $a \to +1$, if $b = 1$ ($\omega = 0$ or $n2\pi$), a branch point at $a = 1$ is encountered and the sum is invalid. For example, if the result above is evaluated for $\omega = 0$ and $a = 1$, the final ratio shown is -1, which suggests that adding positive quantities gives a negative result! The correct way to do all this work is with finite N and then limit $N \to \infty$, and this will be done in more detail for the Change/Hold line code treated in the next section.

We let Maple reduce the ratio where we use $b = e^{i\alpha} = e^{i\omega T_1}$ (ignore right side below for now)

```
b := exp(I*alpha):
f := a*(1+b^2-2*a*b)/((b-a)*(1-a*b)):
g := convert(f,trig):
simplify(g);
```

$$-2\frac{(-\cos(\alpha)+a)\,a}{1+a^2-2\,a\cos(\alpha)}$$

```
d := -p^2*2*(C-a)/(1+a^2-2*a*C);
```

$$d := -2\frac{p^2(C-a)}{1+a^2-2\,a\,C}$$

```
p := (1-a)/2:
d+p: simplify(%): factor(%);
```

$$-\frac{1}{2}\frac{(-1+a)(a+1)(C-1)}{-1-a^2+2\,a\,C}$$

Thus we conclude that

$$\sum_{n=-\infty}^{\infty}\ \sum_{m\neq n}\alpha_{n,m}\left[\,e^{i\omega(m-n)T_1}\right] = (-p^2/a)\,\frac{2a[\cos(\omega T_1)-a]}{1+a^2-2a\cos(\omega T_1)}\sum_{n=-\infty}^{\infty}[1]\ .$$

The curly bracket in (37.9) is then, since $\beta = p$,

$$\{\ldots\} = \left[\ -p^2\,\frac{2\,[\cos(\omega T_1)-a]}{1+a^2-2a\cos(\omega T_1)}\ + p\right]\sum_{n=-\infty}^{\infty}[1]\ .$$

The square bracket is evaluated by Maple as shown on the right above, so using $p = (1-a)/2$ we get

$$\{\ldots\} = \frac{(1/2)(1-a^2)\,[1-\cos(\omega T_1)]}{1+a^2-2a\cos(\omega T_1)}\sum_{n=-\infty}^{\infty}[1] = \frac{(1-a^2)\,[\sin^2(\omega T_1/2)]}{1+a^2-2a\cos(\omega T_1)}\sum_{n=-\infty}^{\infty}[1]$$

where one could write $(1-a^2) = (1-a)(1+a) = 4p(1-p)$ in the numerator. So (37.9) now reads

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,(T_1/T)\,\frac{(1-a^2)\,[\sin^2(\omega T_1/2)]}{1+a^2-2a\cos(\omega T_1)}\sum_{n=-\infty}^{\infty}[1] \qquad a \equiv (1-2p)\ .$$

The sum is either $2\pi\delta(0)$ for the infinite series, or $(2N+1)$ for the finite series. In either case we write the sum as T/T_1 as in box (34.4), and <u>the final result for our AMI line code</u> is

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{(1-a^2)\,[\sin^2(\omega T_1/2)]}{1+a^2-2a\cos(\omega T_1)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p)\ . \qquad (37.10)$$

Using $x = \omega/\omega_1 = fT_1$, this can be written in these alternate forms (the last form uses (33.24)),

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{(1-a^2)\,[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p) \qquad (37.11)$$

$$<P(f)> \;=\; 2\pi\, P_{\mathtt{pulse}}(f)\, \frac{(1-a^2)\,[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p) \qquad (37.11a)$$

$$<P(f)> \;=\; 4p(1-p)\,|\mathcal{X}_{\mathtt{pulse}}(f)|^2\;(1/T_1)\,\frac{\sin^2(\pi f T_1)}{1+(1-2p)^2-2(1-2p)\cos(2\pi f T_1)} \qquad (37.11b)$$

where we recall from the text after (1.4) that $X(\omega) = \mathcal{X}(f)$. This last result agrees with Bennet and Davey (19-123) but they have a leading factor 8 instead of 4. Perhaps this is because they regard the frequency range for f as $(0,\infty)$ instead of $(-\infty,\infty)$ so the left part of the spectrum is folded over to the right side giving them an extra factor of 2.

Note that the AMI spectrum is completely continuous, there is no discrete part at all. The discrete part previously arose from the $\Sigma_n\Sigma_{m\neq n}$ double summation as in (35.9), but here we see no such discrete spectrum generated. It was dispersed into the continuum by the correlation effect between legal bit patterns.

Setting p = 1/2 gives a = 0 and therefore

$$<\mathcal{P}(\omega)> \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega)\;\sin^2(\pi x) \qquad p = 1/2\;\; . \qquad (37.12)$$

Selecting a box of height V and width T_1 we have from (36.1)

$$\mathcal{P}_{\mathtt{pulse}}(\omega) = \frac{|X_{\mathtt{pulse}}(\omega)|^2}{2\pi T_1} = \frac{(T_1 V)^2\,\mathrm{sinc}^2(\pi x)}{2\pi T_1} = V^2\;(T_1/2\pi)\,\mathrm{sinc}^2(\pi x)$$

so that

$$<\mathcal{P}(\omega)> \;=\; V^2(1/\omega_1)\,\mathrm{sinc}^2(\pi x)\,\sin^2(\pi x) \qquad p = 1/2 \qquad x = \omega/\omega_1 \qquad (37.13)$$

$$<P(f)> \;=\; V^2 T_1 \mathrm{sinc}^2(\pi f T_1)\,\sin^2(\pi f T_1) \qquad p = 1/2 \qquad x = fT_1 \qquad (37.13a)$$

This last result agrees with Xiong (2.34) where the code is called AMI-NRZ.

Plot: Ignoring now the overall factor (V^2/ω_1) [or V^2T_1] we get this AMI p = 1/2 power spectrum

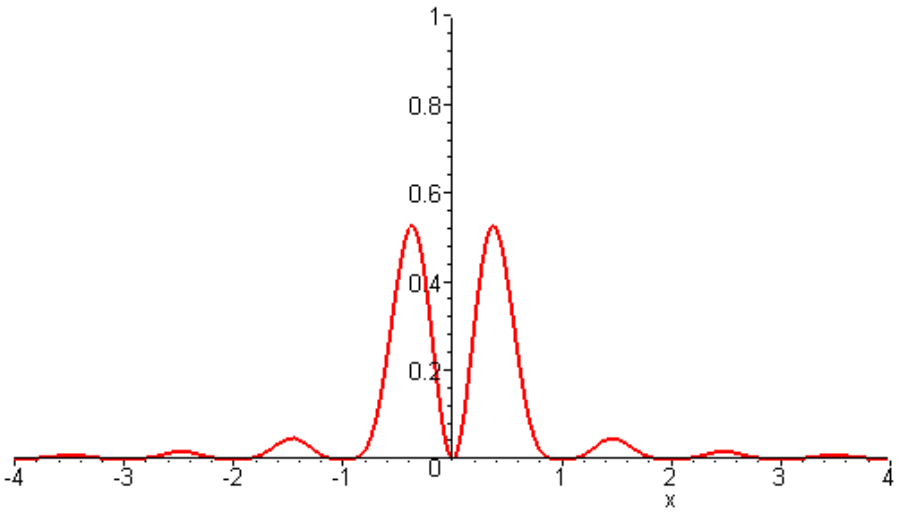


Fig 37.3

This shape is the same as the continuous part of the Manchester spectrum, but the first zero is at 1 instead of 2 since the pulse AMI pulse is twice as wide as the Manchester pulse.

The AMI Limit as p→ 0 (a → +1)

In this limit, we know that our pulse train is just the constant value 0 so $\langle \mathcal{P}(\omega) \rangle = 0$. As noted earlier, our formula is *invalid* in this limit at ω = 0 or 2πn. Still, it is interesting to see what it says:

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega) \frac{(1-a^2)\,[\sin^2(\omega T_1/2)]}{1+a^2-2a\cos(\omega T_1)} = \mathcal{P}_{\mathtt{pulse}}(\omega)\,[\sin^2(\omega T_1/2)]\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)}$$

Using this limit from Appendix A

$$\lim_{a \to +1} (1/\pi)\,\frac{(1-a^2)}{1+a^2-2a\cos(2k)} = \sum_{m=-\infty}^{\infty} \delta(k - m\pi) \qquad \textit{(A.23c)}$$

we find that

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\,[\sin^2(\omega T_1/2)]\,\pi \sum_{m=-\infty}^{\infty} \delta(\omega T_1/2 - m\pi)$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\,\pi \sum_{m=-\infty}^{\infty} [\sin^2(m\pi)]\,\delta(\omega T_1/2 - m\pi) = 0$$

and our expression for $\langle \mathcal{P}(\omega) \rangle$ happens to give the correct answer. The correct expression for $\langle \mathcal{P}(\omega) \rangle$ in this limit of a = +1 (which includes delta spikes including at ω = 0) gives this same answer due to the $\sin^2$ factor. This subject comes up again with the Change/Hold line code in Section 38.

The AMI Limit as p→ 1 (a → -1)

First of all, we can see that in this limit the AMI waveform has alternating-sign pulses. With V = 1, this waveform matches that shown in (34.21),

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega) \sum_{n = \pm odd} \delta(x - m/2) \qquad x = \omega/\omega_1 \quad . \qquad (34.21)$$

Somehow in this limit, the all-continuous AMI spectrum becomes all-discrete! How exactly does this happen? Consider again our continuous AMI result,

$$<\mathcal{P}(\omega)> \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega) \frac{(1-a^2)\,[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \quad . \qquad (37.14)$$

It seems *possible* that this becomes discrete because when a = -1, $(1-a^2) = 0$ and $<\mathcal{P}(\omega)> = 0$ except possibly at singular points where the denominator vanishes. In Appendix A it is shown that

$$\lim_{a\to -1} \delta_8(k, a) = \lim_{a\to -1} (1/\pi) \frac{(1-a^2)\,[\sin^2(k)]}{1+a^2-2a\cos(2k)} = \sum_{m = \pm odd} \delta(k-m\pi/2) \quad . \qquad (A.25a)$$

Therefore we may write

$$<\mathcal{P}(\omega)> \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega)\, \pi\, \delta_8(\pi x, a)$$

$$\rightarrow \mathcal{P}_{\mathtt{pulse}}(\omega)\, \pi \sum_{m = \pm odd} \delta(\pi x - m\pi/2) = \mathcal{P}_{\mathtt{pulse}}(\omega) \sum_{m = \pm odd} \delta(x - m/2)$$

and this agrees with our expected result shown just above.

The coefficient averaging done in this Section for the AMI line code spectrum is based on the excellent discussion of Bennett and Davey p338-341.