38. The Change/Hold Line Code

There are surely more efficient ways to compute the results given below, but the method shown provides a good exercise in directly calculating $\langle y_m y_n \rangle$. The method is very similar to that used for the AMI line code in Section 37. At the end, we apply the results to the NRZI line code.

Pulse Shape:

The pulse shape is the same as for unipolar NRZ,

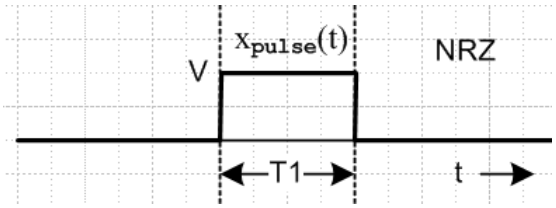


Fig 36.2

$$X_{pulse}(\omega) = (VT_1)\,\mathrm{sinc}(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = V^2(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)\,. \qquad (36.1)$$

In place of amplitude V we will have A and B as described below.

Coding:

The coding uses two amplitudes A and B. Hold or Change coding means that a 0 is encoded as no change (a hold) in the pulse amplitude (it remains what it was), while a 1 is encoded as a change A↔B. Here is an example starting with an A pulse:

```
data   =     [ 1  0  1  1  0  0  1]
encode =  [ A  B  B  A  B  B  B  A]
```

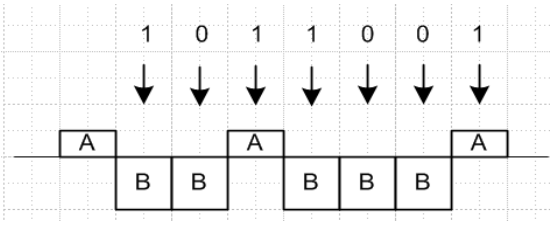


Fig 38.1

Again we shall assume an arbitrary pulse shape and insert the box-pulse at the end.

Coefficients $\alpha_{m,n}$ and β:

The spectrum has the same form given in (37.1) for the AMI code,

$$\langle \mathcal{P}(\omega) \rangle = T_1 \mathcal{P}_{pulse}(\omega)\,(1/T)\,\{ \sum_{n=-\infty}^{\infty} \sum_{m \neq n} \alpha_{n,m} [\, e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} . \qquad (37.1)$$

where $\alpha_{n,m} \equiv \langle y_m y_n \rangle$ and $\beta \equiv \langle y_n^2 \rangle = \alpha_{m,m}$

and our task is now to compute $\alpha_{n,m}$ and β for the Hold/Change line code.

First consider

$$\langle y_n^2 \rangle = [q]A^2 + [(1-q)]\,B^2 ,$$

where q is the probability that slot n has $y_n = A$. In this code, since only change and hold are coded, there is no preference for either amplitude, so q = 1/2 and

$$\langle y_n^2 \rangle = (A^2+B^2)/2 \ . \qquad (38.1)$$

Turning to $\langle y_n y_m \rangle$, consider this picture similar to that used for the AMI case, where the gap is kT_1 units. Here we arbitrarily drawn A > 0 and B < 0 and we draw the pulse as square, but it could be any shape and A and B can have any signs.

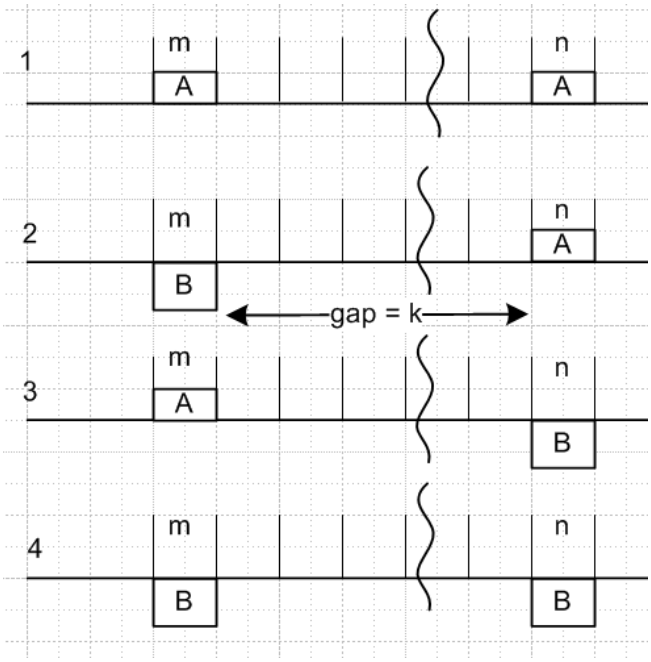


Fig 38.2

Denote the four probabilities as $p^{(AA)}_k$ and so on. Since slot m and slot n must each be filled with either an A or a B, this picture shows the only four possibilities, so (different scaling relative to AMI analysis)

$$p^{(AA)}_k + p^{(AB)}_k + p^{(BA)}_k + p^{(BB)}_k = 1 \ .$$

Note that $p^{(AA)}_k$ is the probability of slot m and slot n both having amplitude A in the statistical pulse train. With these probabilities, we will have

$$\langle y_m y_n \rangle = p^{(AA)}_k \, AA + p^{(AB)}_k \, AB + p^{(BA)}_k \, BA + p^{(BB)}_k \, BB \ . \tag{38.2}$$

In case 1, there are a certain number of holds and changes during the gap such that the overall effect is a hold. The number of changes must have been even. But this same statement can be made about case 4, so cases 1 and 4 have the same probability of existing in the pulse train. Similarly, cases 2 and 3 have the same probability and in those cases the number of changes must be odd. So now we have two variables to worry about and they add to 1/2 :

$$p^{(AA)}_k + p^{(AB)}_k = 1/2 \tag{38.3}$$

$$\langle y_m y_n \rangle = p^{(AA)}_k \, AA + p^{(AB)}_k \, AB + p^{(AB)}_k \, BA + p^{(AA)}_k \, BB$$

$$= p^{(AA)}_k \, (AA + BB) + p^{(AB)}_k \, (AB + BA)$$

so

$$\langle y_m y_n \rangle = p^{(AA)}_k (A^2 + B^2) + p^{(AB)}_k \, 2AB \ . \tag{38.4}$$

If the gap is zero, what is the probability of having an adjacent AA in the pulse stream? The probability of having the left A is 1/2, and the probability for an A being followed by a A is 1-p. Therefore

$$p^{(AA)}_0 = (1/2)(1-p) \ . \tag{38.5}$$

Consider now the gap as shown at value k. We claim that

$$p^{(AA)}_{k+1} = p^{(AA)}_k \, (1-p) + p^{(AB)}_k \, p \ . \tag{38.6}$$

Proof: If it was an AA to start with gap k, then to be AA with gap k+1 we have to add another A which has probability (1-p) since this is a hold. Conversely, if it was an AB we have to add an A which is a change, which has probability p. Then from (38.3),

$$p^{(AA)}_{k+1} = p^{(AA)}_k \, (1-p) + (1/2 - p^{(AA)}_k \,) \, p \ . \tag{38.7}$$

To simplify notation, let $X_k \equiv p^{(AA)}_k$ so that $p^{(AB)}_k = 1/2 - X_k$. Then (38.4) and (38.7) become

$$\langle y_m y_n \rangle = X_k \, (A^2 + B^2) + (1/2 - X_k) 2AB = (A-B)^2 X_k + AB \tag{38.8}$$

$$X_{k+1} = X_k \, (1-p) + (1/2 - X_k) \, p = (1-2p) X_k + p/2$$
$$= a X_k + p/2 \qquad a \equiv 1-2p \ . \tag{38.9}$$

Maple solves this recursion equation as follows, using (38.5) that $X_0 = (1/2)(1-p)$,

```
p := (1-a)/2:
Xk := rsolve({ X(k+1) = (1-2*p)*X(k) + p/2, X(0) = (1-p)/2}, X):
simplify(%);
```

$$\frac{1}{4}a^{(k+1)}+\frac{1}{4}$$

so we find that

$$X_k = (1 + a^{k+1})/4 = p^{(AA)}{}_k \quad . \tag{38.10}$$

As a check, suppose p = 0 so there can be no changes. Then a = 1 and $p^{(AA)}{}_k = 1/2$. We can now have only case 1 or case 4, so we know $p^{(AB)}{}_k = 0$, and that is consistent with $p^{(AA)}{}_k + p^{(AB)}{}_k = 1/2$.

Continuing from (38.8),

$$\langle y_m y_n \rangle = (A-B)^2 X_k + AB = (A-B)^2(1 + a^{k+1})/4 + AB$$

$$= [\,(A-B)^2/4]\, a^{k+1} + (A-B)^2/4 + AB$$

$$= [\,(A-B)^2/4]\, a^{k+1} + [(A+B)^2/4]$$

$$= (1/4)\,[\,(A-B)^2\, a^{k+1} + (A+B)^2\,] \tag{38.11}$$

and we note that the result is indeed symmetric under A↔ B. Again for p = 0 (a=1) we find that $\langle y_m y_n \rangle = (A^2+B^2)/2$ which is the same then as $\langle y_n^2 \rangle$. For a constant pulse train, the amount of slot separation makes no difference.

Since k = |m-n| - 1 in general, we get these final results,

$$\alpha_{n,m} = \langle y_m y_n \rangle = (1/4)\,[\, a^{|m-n|}\,(A-B)^2 + (A+B)^2\,] \qquad a \equiv 1-2p$$

$$\beta = \langle y_n^2 \rangle = (A^2+B^2)/2 \quad . \tag{38.12}$$

To save space, define

$$c \equiv (A-B)^2/4 \qquad d \equiv (A+B)^2/4 \tag{38.13}$$

so then

$$\alpha_{n,m} = \langle y_m y_n \rangle = c\, a^{|m-n|} + d \quad . \tag{38.14}$$

For later use, notice that

$$\beta - d = c \quad . \tag{38.15}$$

Spectrum:

The spectrum is determined by (37.1) quoted above which we repeat here ,

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \sum_{n=-\infty}^{\infty} \sum_{m \neq n} \alpha_{n,m} [e^{i\omega (m-n) T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} . \qquad (37.1)$$

As in the AMI case, the β term in {...} is just

$$\beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] = \beta \sum_{n=-\infty}^{\infty} [1] . \qquad (38.16)$$

Since $\alpha_{n,m} = c\, a^{|m-n|} + d$, the first double sum has two terms. Again looking at the AMI case, we find that the first term is given by, again using $b \equiv e^{i\omega T_1}$,

$$c \sum_{n=-\infty}^{\infty} \sum_{m \neq n} a^{|m-n|} b^{(m-n)} = c \frac{a(1+b^2-2ab)}{(b-a)(1-ab)} \sum_{n=-\infty}^{\infty} [1] . \qquad |a| < 1 \qquad (38.17)$$

To get the second term in the first double sum, we *cannot* just replace c by d and then set a = 1 to get

$$d \frac{a(1+b^2-2ab)}{(b-a)(1-ab)} \sum_{n=-\infty}^{\infty} [1] \text{ (with a = 1)} = d \frac{(1+b^2-2b)}{(b-1)(1-b)} \sum_{n=-\infty}^{\infty} [1] = -d \sum_{n=-\infty}^{\infty} [1] . \qquad \text{// wrong}$$

This is because the result is not valid at a = 1 for ω = 2πn. So we have to do this d sum separately:

$$d \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [e^{i\omega (m-n) T_1}] = d \sum_{n=-\infty}^{\infty} \sum_{m \neq n} b^{m-n} .$$

We now omit the d for a while to evaluate this double sum

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} b^{m-n} = \sum_{n=-\infty}^{\infty} [\sum_{m=-\infty}^{\infty} b^{m-n} - \sum_{m=n} b^{m-n}]$$

$$= (\sum_{n=-\infty}^{\infty} b^{-n}) (\sum_{m=-\infty}^{\infty} b^{m}) - \sum_{n=-\infty}^{\infty} [1] . \qquad (38.18)$$

In the first factor, since $b \equiv e^{i\omega T_1}$, we are facing squared delta functions, so we have to back off to finite N in our processing and later take N→∞. We continue, doing this making use Appendix A,

$$= (2\pi \delta_5(\omega T_1, N))^2 - (2N+1) \qquad \text{// from (A.30)}$$

$= (2N+1)\,[\ (\ 2\pi\delta_5(\omega T_1,N)\ )^2/\ (2N+1)\ -\ 1\ ]$

$= (2N+1)\,[\ (2\pi\delta_6(\omega T_1,N)\ -\ 1\ ]\ .$ // from (A.20)

Then we return to N = ∞ and use

$$\lim_{N\to\infty}\delta_6(k,N) = \sum_{m=-\infty}^{\infty}\delta(k-2\pi m) \qquad (A.21)$$

to get

$$\sum_{n=-\infty}^{\infty}\ \sum_{m\neq n}\ [\ e^{i\omega(m-n)T_1}] = [\ \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\ -1\ ]\ \sum_{n=-\infty}^{\infty}[\ 1\ ] \quad . \qquad (38.19)$$

The -1 is what one gets from the limit a→1 of $\frac{a(1+b^2-2ab)}{(b-a)(1-ab)}$, but we see that there is more.

We can now reinstall d and assemble the pieces to get

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{pulse}(\omega)\,(T_1/T)\ \ \{\ c\,\frac{a(1+b^2-2ab)}{(b-a)(1-ab)} + \beta\ - d\ +\ d\ \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\ \}\ \sum_{n=-\infty}^{\infty}[\ 1\ ]\ .$$

(38.17) (38.16) (38.19) times d

Recalling (38.15) that β - d = c,

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{pulse}(\omega)\,(T_1/T)\ \ \{\ c\,[\,\frac{a(1+b^2-2ab)}{(b-a)(1-ab)}\ +1]\ +\ d\ \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\ \}\ \sum_{n=-\infty}^{\infty}[\ 1\ ]\ .$$

Using the fact (33.22) that $\Sigma_{n=-\infty}^{\infty}[1] = T/T_1$ for both the infinite and finite pulse train cases, we get

$$\langle\mathcal{P}(\omega)\rangle\ =\ \mathcal{P}_{pulse}(\omega)\ \{\ c\,[\,\frac{a(1+b^2-2ab)}{(b-a)(1-ab)}\ +1]\ +\ d\ \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\ \} \qquad (38.20)$$

$a = (1-2p)$ $b = e^{i\omega T_1}$ $c = (A-B)^2/4$ $d = (A+B)^2/4$.

Maple now computes the square-bracketed (sb) expression:

```
b := exp(I*w*T1):
c := (A-B)^2/4:
sb := 1 + a*(1+b^2-2*a*b)/((b-a)*(1-a*b)):
sb1 := convert(sb,trig):simplify(%);
```

$$\frac{-1+a^2}{2\,a\,\cos(w\,T1)-1-a^2}$$

so that

$$\left[\frac{a(1+b^2-2ab)}{(b-a)(1-ab)}+1\right] = \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)}$$

and then here is the final result for the Change/Hold line code :

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \left\{ \left[\frac{(A-B)}{2}\right]^2 \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \left[\frac{(A+B)}{2}\right]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m) \right\} \tag{38.21}$$

where a = (2p-1) and p is the probability of a change while 1-p is the probability of a hold.

We shall now investigate various limits of this result (a confidence-building measure).

Limit A → B: Setting A = B in (38.21) gives

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \left\{ A^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m) \right\} \quad . \tag{38.22}$$

In the case that the pulse is a box of unit height we use (34.22),

$$\mathcal{P}_{\mathbf{pulse}}(\omega) = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2) \tag{38.23}$$

to get

$$\begin{aligned} \langle \mathcal{P}(\omega) \rangle &= A^2\, \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m) \\ &= A^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,(1/\omega_1)\,\mathrm{sinc}^2(\pi m) = A^2\, 2\pi\,\delta(\omega T_1)\,(1/\omega_1) \\ &= A^2\,\delta(\omega) \quad . \end{aligned} \tag{38.24}$$

This is exactly what we expect when A = B, since the pulse train is then just a constant value A !

Limit p→1 (a → -1) :

In this limit we expect to get a square-wave pulse train with alternating values A and B. We make use of this limit from Appendix A with 2k = ωT_1,

$$\lim_{a \to -1} \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} = \pi \sum_{m=\pm\mathrm{odd}} \delta(\omega T_1/2 - m\pi/2) = \sum_{m=\pm\mathrm{odd}} 2\pi\,\delta(\omega T_1 - m\pi) \tag{A.25b}$$

and then the Change/Hold spectral power density (38.21) becomes

$$\langle \mathscr{P}(\omega) \rangle \;=\; \mathscr{P}_{\mathtt{pulse}}(\omega)\;\{\;\left[\frac{(A-B)}{2}\right]^2 \sum_{m\,=\,\pm odd} 2\pi\,\delta(\omega T_1 - m\pi) \;+\; \left[\frac{(A+B)}{2}\right]^2 \sum_{m\,=\,-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\;\}\;.$$

Installing from (34.22) the unit-height box pulse shape $\mathscr{P}_{\mathtt{pulse}}(\omega) = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)$ and using (38.24) the second term becomes just $\left[\frac{(A+B)}{2}\right]^2 \delta(\omega)$ while the first term is

$$\left[\frac{(A-B)}{2}\right]^2 \sum_{m\,=\,\pm odd} 2\pi\,\delta(\omega T_1 - m\pi)\,(1/\omega_1)\,\mathrm{sinc}^2(m\pi/2)$$

$$= \left[\frac{(A-B)}{2}\right]^2 2\pi \sum_{m\,=\,\pm odd} \delta(\omega T_1 - m\pi)\,(1/\omega_1)\,(m\pi/2)^{-2}$$

$$= (A-B)^2\;(1/\pi^2) \sum_{m\,=\,\pm odd} (1/m^2)\,\delta(\omega - m\omega_1/2)$$

giving a final result,

$$\langle \mathscr{P}(\omega) \rangle \;= \left[\frac{(A-B)}{\pi}\right]^2 \sum_{m\,=\,\pm odd} (1/m^2)\,\delta(\omega - m\omega_1/2) \;+ \left[\frac{(A+B)}{2}\right]^2 \delta(\omega)\;. \qquad (38.25)$$

Comparing the first term with (34.23), we see that it is the spectrum of a square wave whose peak-to-peak amplitude is (A-B), which is exactly what it should be. The second term then correctly accounts for the expected average DC level of (A+B)/2.

Limit $p \to 0$ ($a \to +1$) :

In this case for a square wave we expect to get a result appropriate for an ensemble of pulse trains half of which have constant value A and the other have constant value B, since all pulse trains are in a permanent hold state with p = 0; nothing changes. This time we use this limit (A.23c) with $2k = \omega T_1$,

$$\lim_{a \to +1} \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} \;= \pi \sum_{m\,=\,-\infty}^{\infty} \delta(\omega T_1/2 - m\pi) \;= \sum_{m\,=\,-\infty}^{\infty} 2\pi\delta(\omega T_1 - m2\pi) \qquad \textit{(A.23c)}$$

to get from (38.21),

$$\langle \mathscr{P}(\omega) \rangle \;=\; \mathscr{P}_{\mathtt{pulse}}(\omega)\;\{\;\left[\frac{(A-B)}{2}\right]^2 \sum_{m\,=\,-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2m\pi) \;+\; \left[\frac{(A+B)}{2}\right]^2 \sum_{m\,=\,-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\;\}$$

$$= \frac{A^2+B^2}{2}\;\mathscr{P}_{\mathtt{pulse}}(\omega) \sum_{m\,=\,-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2m\pi)\;. \qquad (38.26)$$

Ignoring the leading factor, this agrees with the first line of (33.25) which was developed for a simple pulse train with unit amplitudes $y_n = 1$. This result then describes the average spectral power of an ensemble in which 50% of the pulse trains have amplitude A and the rest amplitude B.

Installing the square pulse spectrum (38.23) $\mathcal{P}_{\mathtt{pulse}}(\omega) = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)$ gives

$$\langle \mathcal{P}(\omega) \rangle = \frac{A^2+B^2}{2}\,(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2) \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2m\pi)$$

$$= \frac{A^2+B^2}{2}\,(1/\omega_1)\,2\pi\,\delta(\omega T_1) = \frac{A^2+B^2}{2}\,(T_1\omega_1)^{-1}\,2\pi\,\delta(\omega)$$

$$= \frac{A^2+B^2}{2}\,\delta(\omega)$$

which describes an ensemble of constant pulse trains 50% of which are x(t) = A and the rest x(t) = B.

Limit p→1/2 (a → 0)

Recall again the general result (38.21),

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\,\{\ [\frac{(A-B)}{2}\ ]^2\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}\ ]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ \}\ . \qquad \textit{(38.21)}$$

The ratio becomes unity so the spectral power density is then

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\,\{\ [\frac{(A-B)}{2}\ ]^2 + [\frac{(A+B)}{2}\ ]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ \}\ . \qquad (38.27)$$

Box-Shaped Pulse for general p: Start as just above with (38.21) and insert $\mathcal{P}_{\mathtt{pulse}}(\omega)$ for the box,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathtt{pulse}}(\omega)\,\{\ [\frac{(A-B)}{2}\ ]^2\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}\ ]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ \}$$

$\mathcal{P}_{\mathtt{pulse}}(\omega) = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)$.

As usual, the second term becomes $[\frac{(A+B)}{2}\ ]^2\,\delta(\omega)$, so the result is [a = 1-2p, x = $\omega/\omega_1 = fT_1$]

$$\langle \mathcal{P}(\omega) \rangle = [\frac{(A-B)}{2}\ ]^2\,(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + (1/\omega_1)[\frac{(A+B)}{2}\ ]^2\,\delta(\omega/\omega_1) \qquad (38.28)$$

$$\langle P(f) \rangle = [\frac{(A-B)}{2}\ ]^2\,T_1\,\mathrm{sinc}^2(\pi f T_1)\,\frac{(1-a^2)}{1+a^2-2a\cos(2\pi f T_1)} + [\frac{(A+B)}{2}\ ]^2\,\delta(f)\ . \qquad (38.28a)$$

Box-Shaped Pulse for p = 1/2 (a = 0):

$$\langle \mathcal{P}(\omega) \rangle = [\frac{(A-B)}{2}\ ]^2\,(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2) + [\frac{(A+B)}{2}\ ]^2\,\delta(\omega)$$

(38.29)

$$= (1/\omega_1)\ \{\ \ [\frac{(A-B)}{2}]^2\ \mathrm{sinc}^2(\pi x)\ \ +\ \ [\frac{(A+B)}{2}]^2\ \delta(x)\ \ \}\ \ \ x \equiv \frac{\omega}{\omega_1}\ \ = f\,T_1$$

$$\langle P(f) \rangle = T_1\,[\frac{(A-B)}{2}]^2\ \mathrm{sinc}^2(\pi f T_1)\ \ +\ \ [\frac{(A+B)}{2}]^2\ \delta(f)\ \ \ . \qquad (38.29a)$$

Ignoring the factor $(1/\omega_1)$ in (38.29), we make this plot of $\langle \mathscr{P}(\omega) \rangle$, which is the same as for unipolar NRZ but with different scaling factors for the two terms,

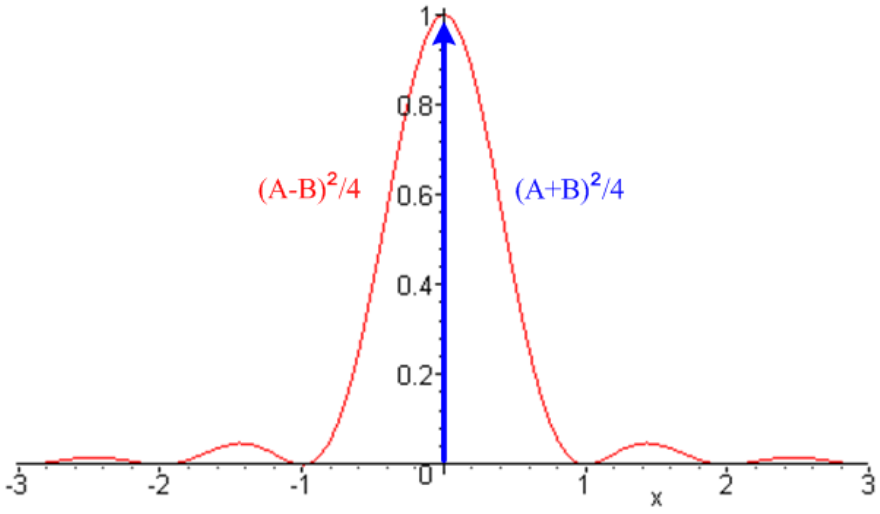


Fig 38.3

Example 1: Unipolar NRZI line code

Coding: This is a special case of Change/Hold encoding where A = 1 and B = 0.

```
data   =      [ 1   0   1   1   0   0   1]
encode =   [ 1   0   0   1   0   0   0   1]
```

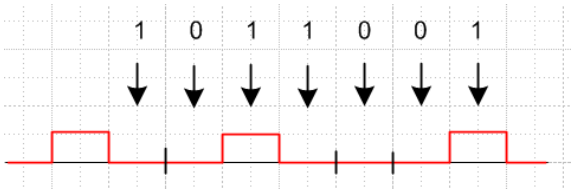


Fig 38.4

NRZI means NRZ Invert-on-1, where NRZ means non-return to zero (see comments at the start of Section 36). NRZI does not mean "NRZ inverted". Some other sources use A = 0 and B = 1 so then transitions happen on 0 instead of 1, as in the standard for USB (Universal Serial Bus). The Change/Hold spectra are symmetric in A↔B, so the NRZI spectra are the same for either convention.

Coefficients $\alpha_{m,n}$ and β:

$$\alpha_{n,m} = \langle y_m y_n \rangle = (1/4)\,[\, a^{|m-n|}\,(A-B)^2 + (A+B)^2\,] = (1/4)\,[\, a^{|m-n|} + 1\,]$$

$$\beta = \langle y_n^2 \rangle = (A^2+B^2)/2 = 1/2 \qquad (38.30)$$

Spectrum: From (38.21), and with a = (1-2p),

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\{\,[\,\frac{(A-B)}{2}\,]^2\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\,\frac{(A+B)}{2}\,]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\,\{\,\frac{1}{4}\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \frac{1}{4}\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\}\,. \qquad (38.31)$$

Box-Shaped Pulse for general p: From (38.28),

$$\langle \mathcal{P}(\omega) \rangle = \frac{1}{4}\,(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \frac{1}{4}\,\delta(\omega)\,. \qquad (38.32)$$

Box-Shaped Pulse for p = 1/2 (a = 0): From (38.29),

$$\langle \mathcal{P}(\omega) \rangle = (1/\omega_1)\,\{\,\frac{1}{4}\,\mathrm{sinc}^2(\pi x) + \frac{1}{4}\,\delta(x)\,\} \qquad x \equiv \frac{\omega}{\omega_1} = fT_1 \qquad (38.33)$$

$$\langle P(f) \rangle = T_1\,\frac{1}{4}\,\mathrm{sinc}^2(\pi x) + \frac{1}{4}\,\delta(f)\,. \qquad (38.33a)$$

The plot is that of Fig 38.3, but with each factor being 1/4.

This spectrum is exactly the same as that for unipolar NRZ shown in (36.3b) with V = 1. One way to understand this fact is that for every NRZ sequence y_n there is a NRZI sequence y'_n given by (36.3).

$$y'_n = y_n - y_{n-1} \quad . \qquad \text{// mod-2 math}$$

This equation can be solved for y_n in terms of y'_n (assume $y_0 = 0$),

$$y_n = \Sigma_{m=1}^{n}\, y'_m \quad n = 1,2,3....$$

Consider the space of all random sequences of 1's and 0's (random pulse trains p = 1/2). Since we just showed that the relation $\{y_n\} \leftrightarrow \{y'_n\}$ is one-to-one, the mapping f: $\{y_n\} \rightarrow \{y'_n\}$ just reorders the set of random sequences in the ensemble used to compute the spectral power density, so that density cannot change.

In contrast, for $p \neq 1/2$, the NRZI power spectrum (38.32) is quite different from the NRZ power spectrum (36.3),

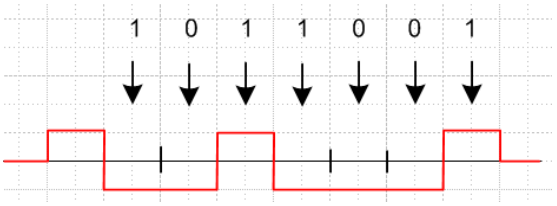
$$\langle \mathcal{P}(\omega) \rangle = \frac{1}{4}\,(1/\omega_1)\left[\,\mathrm{sinc}^2(\pi x)\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \delta(x)\right] \qquad \text{// unipolar NRZI} \qquad \textit{(38.32)}$$

$$a = (2p-1)$$

$$\langle \mathcal{P}(\omega) \rangle == \frac{1}{4}\,(1/\omega_1)\left[\mathrm{sinc}^2(\pi x)\,(1-a^2) + (2p)^2\,\delta(x)\right] \qquad \text{// unipolar NRZ} \qquad \textit{(36.3)}$$

Example 2: Bipolar NRZI line code

```
data   =      [ 1   0   1   1   0   0   1]
encode =   [ 1 -1 -1   1 -1 -1 -1   1]
```



In this case A = 1 and B = -1 so the general NRZI spectrum (38.21) becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} \qquad a = (2p-1) \qquad \text{// bipolar NRZI} \qquad (38.34)$$

and for p = 1/2 (a=0) we obtain,

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \; . \qquad \text{// } = (1/2\pi)\, T_1\, \mathrm{sinc}^2(\omega T_1/2) \text{ for the box pulse (36.1)} \qquad (38.35)$$

In contrast, the Bipolar NRZ spectrum is $\mathcal{P}_{\mathbf{pulse}}(\omega)\,[(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\,]$ where $\alpha = (1-2p)^2$, $\beta = 1$, and $(\beta-\alpha) = 4p(1-p) = 1-a^2$ so it's spectrum is

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\,[(1-a^2) + (1-2p)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\,] \quad \text{// bipolar NRZ} \qquad (38.36)$$

If p = 1/2 (a=0) then the Bipolar NRZ spectrum becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \qquad (38.37)$$

which is the same as for Bipolar NRZI.