Power Formulas for Finite Pulse Trains PhL 8.2.13

Here is where I realized that WK also works for finite pulse trains. I added all this into FT, but it is split into two places inside FT.

Question #1: What does WK say for a finite pulse train?

In FT I keep assuming infinite pulse trains at critical points, but I would like the finite N formulas as well. Let's start on the R"(z) side of the docket. I define

$$r_s \equiv \lim_{N\to\infty}\left[\frac{1}{(2N+1)}\sum_{n=-N}^{N} y_n\, y_{n+s}\right] \equiv \langle y_n\, y_{n+s}\rangle_1 . \qquad (32.16)$$

But for finite N and pulse train of length 2N+1, which has only y_{-N} .... y_0....y_N we get

$$r_s = \frac{1}{(2N+1)}\sum_{n=-N}^{N} y_n\, y_{n+s}$$

where terms vanish that maybe I can make explicit this way

$$r_s = \frac{1}{(2N+1)}\sum_{n=-N}^{N} y_n\, y_{n+s}\ \Theta(-N \le n+s \le N) = \frac{1}{(2N+1)}\sum_{n=-N}^{N} y_n\, y_{n+s}\ \Theta(-N-n \le s \le N-n)$$

For n = N we find $\Theta(-N-N \le s \le N-N) = \Theta(-2N \le s \le 0)$ so r_s vanishes when s < -2N. and similarly on the high side, we I think we can say

r_s is non-vanishing only in the range -2N to +2N.

Now we go on to the next step: compute R"(z). We can still use ∞ endpoints with the understanding that r_s vanishes out of range either way. And the same for y_n etc. So copy and paste to get

$$R"(z) \equiv \sum_{s=-\infty}^{\infty} r_s\, z^{-s} = \sum_{s=-\infty}^{\infty}\left\{\frac{T_1}{T}\sum_{n=-\infty}^{\infty} y_n\, y_{n+s}\right\} z^{-s} = \frac{T_1}{T}\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\left[y_n\, z^n\right]\left[y_{n+s}\, z^{-(n+s)}\right]$$

$$= \frac{T_1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[y_n\, z^n\right]\left[y_m\, z^{-m}\right] \qquad m \equiv n+s$$

$$= \frac{T_1}{T}\left[\sum_{n=-\infty}^{\infty} y_n\, z^n\right]\left[\sum_{m=-\infty}^{\infty} y_m\, z^{-m}\right] = \frac{T_1}{T}\, Y"(z)^*\, Y"(z)$$

So the result continues to be valid for our finite pulse train where $T = (2N+1)T_1$ and

$$Y''(z) = \sum_{n=-N}^{N} y_n z^{-n}$$

Conclusion: for finite pulse train (-N,N), we can still claim WK which says

$$R''(z) = \sum_{s=-2N}^{2N} r_s z^{-s} = (2N+1)^{-1} Y''(z)^* Y''(z) \qquad Y''(z) = \sum_{n=-N}^{N} y_n z^{-n}$$

Question #2: Is the power formula valid for a finite pulse train?

Look at summary box (34.4). First we have

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T}$$

which is certainly valid for any pulse train, finite or not. Then we know that

$$X(\omega) = (1/T_1) X_{pulse}(\omega)\, Y'(\omega) = X_{pulse}(\omega)\, Y''(z)$$

so everything "goes through" for a finite pulse train, in particular

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2 \, |\, Y''(z)\, |^2$$

and

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \, |\, Y''(z)\, |^2$$

where now the ratio is $(2N+1)^{-1}$.

Then we apply the above WK to get

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$$

Conclusion: This power formula IS VALID for a finite pulse train.