Clarification of finite N tale of two formulas PhL 8.6.13

More issues now dealt with in the rewritten (as of Aug 8) Section 35.

(1) In an older FT this was the dichotomy I presented:

(1) one pulse train / autocorrelation, and // see black text below at the start
(2) ensemble.

In the current FT, I describe a completely different dichotomy,

(1) autocorrelation method
(2) double sum method

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left(\sum_{s=-\infty}^{\infty} r_s z^{-s} \right)$$ first line of (35.6) "autocorrelation"

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} y_m z^{-m} \right)^* \left(\sum_{n=-\infty}^{\infty} y_n z^{-n} \right)$$ (35.26)

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_m y_n z^{m-n} \right)$$ third line of (35.6) "double sum"

So what happened to the older dichotomy? In Section 35 I have sliced the pie differently. I have separate sections for "one pulse train" and "ensemble of pulse trains". I do show in the infinite case that the single pulse train and ensemble have the same statistics, such as $\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$. In fact I have a whole little section on that. For the infinite case, both methods of my first dichotomy always give the same result, so there is no point in computing things both ways (as I once was doing in AMI and Change/Hold). Thus I no longer present that as a dichotomy.

(2) Below I discovered the "gray region" for the first time.

What does stationarity say for my two formulas?

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s}$$ one pulse train $z \equiv e^{i\omega T_1}$ (35.6)

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$ ensemble of pulse trains *(35.5)*

It is only a property of $\langle y_m y_{m+s} \rangle$, not of $\langle y_m y_{m+s} \rangle_1$ which is always "stationary" by its definition. So in the second equation, stationarity says

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \left(\frac{T_1}{T} \sum_{m=-\infty}^{\infty} [1] \right)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$

So with stationarity, here are our two formulas:

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} \qquad \text{one pulse train}$$

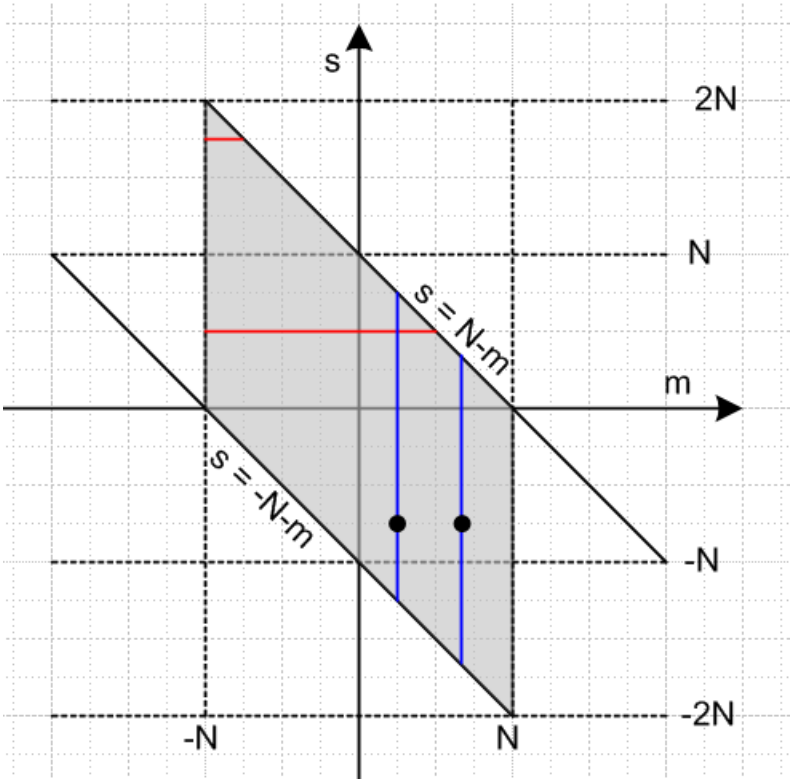
$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$

So this makes pretty clear that for infinite pulse trains, you don't have to do things by "both methods" because they will always be the same. So I need to clean up AMI and Change Hold in that regard.

What happens if the sums are finite??? This is the big question. For a single finite pulse train,

$$y_m y_{m+s} = y_m y_{m+s}\, \theta(m < N)\theta(m > -N)\, \theta(m+s < N)\theta(m+s > -N)$$

which is non-vanishing only over the finite gray region shown below



Now consider the type 1 average over a pulse train

$$\langle y_m y_{m+s} \rangle_1 = \frac{1}{2N+1} \sum_{m=-N}^{N} y_m y_{m+s}$$

The sum for two positive values of s is shown in the picture as a red line. There are many fewer terms in the sum shown by the upper red line which is for some s close to 2N, so one would expect $\langle y_m y_{m+s} \rangle_1$ to be smaller for the upper value of s compared with the lower value of s. But $\langle y_m y_{m+s} \rangle_1$ is always independent of m, and we write $r_s = \langle y_m y_{m+s} \rangle_1$.

Now consider the ensemble average $\langle y_m y_{m+s} \rangle$.

$$\langle y_m y_{m+s} \rangle = \frac{1}{I} \sum_{i=1}^{I} y_m^{(i)} y_{m+s}^{(i)}$$

We know that $y_m^{(i)} y_{m+s}^{(i)}$ is non vanishing only in the gray area shown above. Imagine now extruding the above picture out of the plane of paper, so we then have a separate (but identical) gray area for each pulse train i. Then the ensemble Σ_i sum here is a sum perpendicular to the plane of paper and is at a fixed point in the (m,s) plane. For some particular value of m, such as the left blue line, $\langle y_m y_{m+s} \rangle$ will be non-vanishing for all s corresponding to the blue line. For a larger value of m, we get the right blue line which has a different range of s. Nothing in this picture precludes the possibility that $\langle y_m y_{m+s} \rangle$ be the same for these two values of m and the same value of s, as suggested by the two black dots, each of which represents an ensemble sum perpendicular to the plane of paper.

Conclusion: The assumption of stationarity can be applied to both infinite and finite pulse trains. Nothing blocks the *possibility* for this assumption when dealing with a finite pulse train.

Now back to the two formulas:

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} \qquad \text{one pulse train} \quad z \equiv e^{i\omega T_1} \qquad (35.6)$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \qquad \text{ensemble of pulse trains} \qquad (35.5)$$

For a finite pulse train we can write these as

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle_1 z^{-s} \qquad \text{one pulse train} \quad z \equiv e^{i\omega T_1} \qquad (35.6)$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \sum_{s=-2N}^{2N} \sum_{m=-N}^{N} \langle y_m y_{m+s} \rangle z^{-s} \qquad \text{ensemble of pulse trains}$$

(35.5)

With the stationarity assumption, the lower equation becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{1}{2N+1} \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle z^{-s} \sum_{m=-N}^{N} [1]$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle z^{-s}$$

so with the stationarity assumption our two formulas become

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle_1 z^{-s} \qquad \text{one pulse train} \qquad z \equiv e^{i\omega T_1} \qquad (35.6)$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle z^{-s} \qquad \text{ensemble of pulse trains}$$

These formulas are not the same. We have

$\langle y_m y_{m+s} \rangle_1 \neq \langle y_m y_{m+s} \rangle$	for small N	$\langle \mathcal{P}(\omega) \rangle \neq \mathcal{P}(\omega)$
$\langle y_m y_{m+s} \rangle_1 \approx \langle y_m y_{m+s} \rangle$	for large N,	$\langle \mathcal{P}(\omega) \rangle \approx \mathcal{P}(\omega)$
$\langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle$	$N = \infty$	$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$

Only in the N=∞ case are the two methods the same.

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I thought I was doing well, but now I am confused again. I am losing the main point. There seem to be two formulas, but the second is just the ensemble average of the first. So you can say

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} r_s z^{-s} = \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) R''(z)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} | Y''(z) |^2$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \langle R''(z) \rangle$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{T_1}{T} \langle | Y''(z) |^2 \rangle$$

It just happens that $r_s = \langle y_m y_{m+s} \rangle_1$ which brings in this funny 1 type average. Stationarity applies only to the second equation which involves Random Variables, and then we have

$$\mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \sum_{m=-\infty}^{\infty} 1$$

$$= \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$

So we then end up with these two equations. The first is exact, the second is exact with the assumption of stationarity

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s}$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s}$$

They are NOT the same equation because they compute different things. It does seem appropriate then to compare the two types of averages as I have done. Now suppose Y_n and Y_m are independent. Then we have

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} = \mathcal{P}_{pulse}(\omega) R''(z) \qquad \text{single pulse train}$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \left[\langle y_m^2 \rangle + \sum_{s \neq 0} \langle y_m \rangle^2 z^{-s} \right] \qquad \text{ensemble of pulse trains}$$

$$= \mathcal{P}_{pulse}(\omega) \left[(\sigma^2 + \mu^2) + \mu^2 \sum_{s \neq 0} z^{-s} \right]$$

Again, for the second line we have assumed both stationarity and independence, whereas in the first line we have assumed nothing at all. Both equations are valid for finite pulse trains as well as ∞ .

What then is the deal with a finite pulse train?

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Whole new question! What is the meaning of r_s ? I write it for a finite sequence this way

$$\langle r_s \rangle = \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} \langle y_m y_{m+s} \rangle$$

using my gray area thing. Where does this normalization factor come from? I *defined* r_s for my own personal use in (32.17) and that is *why* I get $R''(z)=\frac{T_1}{T}\,|\,Y''(z)\,|^2$ and that in turn is *why* I got

$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$. The norm factor must be there to make this all work right.