Section 34 (c) An extended example PhL 3.26.05

This example is fairly general and we shall at the end consider its special cases, since these will be referred to later on.

The example is this: The pulse train is composed of some general pulse $x_{\mathbf{pulse}}(t)$ and the amplitudes are repeated sequences of A,B.

Our starting point is (34.8) with (34.7), where we assume N is large and later we will take $N\to\infty$:

$$|X(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2\,(1/T_1)^2\,|X'(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2\,|\,X''(z)\,|^2 \qquad z = e^{i\omega T_1} \qquad (34.8)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n\, e^{-i\omega n T_1}. \qquad (34.7)$$

The main problem is to compute X"(z) and then square it. We have

$$\sum_{n=-N}^{N} y_n\, e^{-i\omega n T_1} = A \sum_{n\ \text{even}} e^{-i\omega n T_1} + B \sum_{n\ \text{odd}} e^{-i\omega n T_1}$$

Now process the sums as follows, where

$$\sum_{n\ \text{even}} e^{-i\omega n T_1} = \sum_{m=-N/2}^{N/2} e^{-i\omega (2m) T_1} \quad \text{where we used } n = 2m$$

$$\sum_{n\ \text{odd}} e^{-i\omega n T_1} = \sum_{n=(-N-1)/2}^{(N-1)/2} e^{-i\omega (2m+1) T_1} \quad \text{where we used } n = 2m + 1$$

We assume N is very large, so we regard $(N\pm1)/2 \approx N/2$. We then find

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n\, e^{-i\omega n T_1} = [\,A + B\, e^{-i\omega n T_1}] \sum_{m=-N/2}^{N/2} e^{-i\omega (2m) T_1}\,.$$

We now use (13.3),

$$\sum_{n=-N}^{N} e^{\mathbf{i}nk} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N)\ , \qquad -\infty < k < \infty \qquad (13.3)$$

to write

$$\sum_{m=-N/2}^{N/2} e^{-i\omega (2m) T_1} = 2\pi\, \delta_5(2\omega T_1, N/2)$$

where δ_5 and δ_6 to come are explained in Appendix A. Therefore

$$X''(z) = [\, A + B\, e^{-i\omega n T_1}] 2\pi\, \delta_5(2\omega T_1, N/2) \tag{34.16}$$

Squaring we get

$$|X''(z)|^2 = |A + Be^{-i\omega T_1}|^2\, [2\pi\, \delta_5(2\omega T_1, N/2)]^2$$

Then from (A.20) applied with $N \rightarrow N/2$

$$\delta_6(k,N/2) \equiv \frac{1}{2\pi}\ \frac{[2\pi\delta_5(k,N/2)]^2}{(N+1)} \tag{A.20}$$

we get

$$|X''(z)|^2 = |A + Be^{-i\omega T_1}|^2\, [2\pi\, \delta_5(2\omega T_1, N/2)]^2$$

or

$$\frac{|X''(z)|^2}{2\pi(N+1)} = |A + Be^{-i\omega T_1}|^2 \left\{ \frac{1}{2\pi}\ \frac{[2\pi\delta_5(2\omega T_1, N/2)]^2}{(N+1)} \right\} = |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1, N/2)$$

Now for large N we ignore the difference between N and N + 1 and so on, so we divide both sides by 2 to get,

$$\frac{|X''(z)|^2}{2\pi(2N+1)} = (1/2)\, |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1, N/2) \tag{34.17}$$

Notice that a very important factor of 1/2 appears on the right in the last step. We now add back the squared pulse spectrum to get

$$\frac{|X(\omega)|^2}{2\pi(2N+1)} = |X_{pulse}(\omega)|^2\, \frac{|X''(z)|^2}{2\pi(2N+1)} = |X_{pulse}(\omega)|^2\ (1/2)|A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1, N/2) \tag{34.18}$$

If we divide both sides by T_1 the left side is $\frac{|X(\omega)|^2}{2\pi T}$ where T is the length of the pulse train and this in turn equals $\mathcal{P}(\omega)$ all as shown in box (34.4). So for large N we have shown that

$$\mathcal{P}(\omega) = |X_{pulse}(\omega)|^2\, (1/T_1)(1/2)\, |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1, N/2) \tag{34.19}$$

Now at last we take the limit $N \rightarrow \infty$ and use

$$\lim_{N\rightarrow\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m) \tag{A.21}$$

to get our desired infinite pulse train result

$$\mathcal{P}(\omega) = |X_{\mathtt{pulse}}(\omega)|^2 (1/T_1)(1/2) |A + Be^{-i\omega T_1}|^2 \sum_{m=-\infty}^{\infty} \delta(2\omega T_1 - 2\pi m) \qquad (34.20)$$

We then make two ancillary calculations

$$|A + Be^{-i\omega T_1}|^2 = |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)$$

$$\delta(2\omega T_1 - 2\pi m) = (2T_1)^{-1} \delta(\omega - (1/2)[2\pi/T_1]m) = (2T_1)^{-1} \delta(\omega - m\omega_1/2) \qquad \omega_1 = 2\pi/T_1$$

Here then is our final result:

$$\mathcal{P}(\omega) = |X_{\mathtt{pulse}}(\omega)|^2 (1/4)(1/T_1^2) \{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \qquad (34.21)$$

We shall now challenge this result for various special cases.

Special Case 1. $x_{\mathtt{pulse}}(t)$ is a box of width T_1 and height 1. From (9.2) we then have

$$X_{\mathtt{pulse}}(\omega) = T_1 \operatorname{sinc}(\omega T_1/2)$$

so that

$$\mathcal{P}(\omega) = (1/4) \operatorname{sinc}^2(\omega T_1/2) \{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \qquad (34.22)$$

Special Case 1a: Let A = B = 1. First, let's calculate this another way. If the pulse train is a DC signal x(t) = 1, it must be that $\mathcal{P}(\omega) = k\delta(\omega)$ for some k. The total power is then $P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) = k$. But for a load of R = 1Ω, we know that P will be 1 watt for A = 1 volt, so k = 1 and $\mathcal{P}(\omega) = \delta(\omega)$.

So how exactly is this going to happen? First

$$\{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} = 2[1+\cos(\omega T_1)] = 4 \cos^2(\omega T_1/2)$$

Evaluated at $\omega = m\omega_1/2$ we get $\omega T_1/2 = m\pi/2$. So then

$$\mathcal{P}(\omega) = (1/4) \sum_{m=-\infty}^{\infty} \operatorname{sinc}^2(m\pi/2) \{4 \cos^2(m\pi/2)\} \delta(\omega - m\omega_1/2) = \delta(\omega)$$

For all integer $m \neq 0$ we have a factor $2\sin(m\pi/2)\cos(m\pi/2) = \sin(m\pi) = 0$ so there is no contribution. Then for m = 0, sinc = 1 and cos = 1 and the result is $\mathcal{P}(\omega) = \delta(\omega)$, as expected.

Special Case 1b: Let A = 1 and B = 0. Then

$$\mathcal{P}(\omega) = (1/4)\,\mathrm{sinc}^2(\omega T_1/2)\,\{1\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$= (1/4) \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(m\pi/2)\,\delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m\neq 0} (m\pi/2)^{-2}\,\sin^2(m\pi/2)\,\delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m\ \mathrm{odd}} (m\pi/2)^{-2}\,\sin^2(m\pi/2)\,\delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m\ \mathrm{odd}} (m\pi/2)^{-2}\,\delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4)\,(2/\pi)^2 \sum_{m\ \mathrm{odd}} \frac{1}{m^2}\,\delta(\omega - m\omega_1/2) \qquad (34.23)$$

For verification, we recall (33.27) that $\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\,\delta(\omega - m\omega_1)$ and we consider the waveform shown in Fig ** with coefficients as in (16.2a). For that waveform we get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1\tau/T_1)^2\,\mathrm{sinc}^2(m\pi\tau/T_1)\,\delta(\omega - m\omega_1)$$

To get a 50% duty cycle, we set $\tau = T_1/2$ to get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1/2)^2\,\mathrm{sinc}^2(m\pi/2)\,\delta(\omega - m\omega_1)$$

But to get our current A = 1 B = 0 waveform, we have to scale $T_1/2 \rightarrow T_1$ in Fig ** . This means we are taking $T_1 \rightarrow 2T_1$ so that means $\omega_1 \rightarrow \omega_1/2$ and then we get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1/2)^2\,\mathrm{sinc}^2(m\pi/2)\,\delta(\omega - m\omega_1/2)$$

which agrees an early line of our result quoted above. This can be written, by the way, as

$$\mathcal{P}(\omega) = (1/4)\,\delta(\omega) + \sum_{m=\mathrm{odd}} (1/2)^2\,\mathrm{sinc}^2(m\pi/2)\,\delta(\omega - m\omega_1/2)$$

The origin for t = 0 in the waveform plays no role in $\mathcal{P}(\omega)$ since the phase shift dies off in $|X(\omega)|^2$. The DC component is consistent with an average DC level of 1/2.

Special Case 1c: Let A = 1 and B = -1. Then

$$\mathcal{P}(\omega) = (1/4)\,\mathrm{sinc}^2(\omega T_1/2)\,\{\,|A|^2 + |B|^2 + 2|A||B|\cos(\omega T_1)\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$= (1/4)\,\mathrm{sinc}^2(\omega T_1/2)\,\{\,2 - 2\cos(\omega T_1)\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$= \mathrm{sinc}^2(\omega T_1/2)\,\sin^2(\omega T_1/2) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(m\pi/2)\,\sin^2(m\pi/2)\;\delta(\omega - m\omega_1/2)$$

$$= \sum_{m=-\infty}^{\infty} (m\pi/2)^{-2}\,\sin^4(m\pi/2)\;\delta(\omega - m\omega_1/2)$$

$$= \sum_{m=\,\mathrm{odd}} (m\pi/2)^{-2}\,\sin^4(m\pi/2)\;\delta(\omega - m\omega_1/2) = \sum_{m=\,\mathrm{odd}} (m\pi/2)^{-2}\;\delta(\omega - m\omega_1/2)$$

so the final result is

$$\mathcal{P}(\omega) = (2/\pi)^2 \sum_{m=\,\mathrm{odd}} \frac{1}{m^2}\;\delta(\omega - m\omega_1/2) \tag{34.24}$$

Verification: If we add a DC offset of +1 to our case 1c waveform, we will get

$$\mathcal{P}(\omega) = \delta(\omega) + (2/\pi)^2 \sum_{m=\,\mathrm{odd}} \frac{1}{m^2}\;\delta(\omega - m\omega_1/2)$$

But this is the same waveform as in Special Case 1b with A = 2 and B = 0, so $\mathcal{P}(\omega)$ should be 4x the result of Special Case 1b, which in fact it is.

Exercises for the Reader:

(a) If the repeating amplitude sequence is A,B,C then

$$\mathcal{P}(\omega) = |X_{\mathbf{pulse}}(\omega)|^2\,(1/T_1)(1/3)\,|A + Be^{-i\omega T_1} + C\,e^{-i2\omega T_1}|^2 \sum_{m=-\infty}^{\infty} \delta(3\omega T_1 - 2\pi m) \tag{34.25}$$

(b) If the repeating sequence is $A_0, A_1 \ldots A_{M-1}$ then

$$\mathcal{P}(\omega) = |X_{\mathbf{pulse}}(\omega)|^2\,(1/T_1)(1/M)\,\Big|\sum_{k=0}^{M-1} A_k e^{-ik\omega T_1}\Big|^2 \sum_{m=-\infty}^{\infty} \delta(M\omega T_1 - 2\pi m) \tag{34.26}$$