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Appendix C: The Fourier Transform and its relation to the Hilbert Transform

In this Appendix we refer to the Fourier Integral Transform simply as the Fourier transform. We develop more "facts" about Fourier transforms, including new notations, and present a set of closely related examples. The pf pseudofunction and principal part integrals are introduced in the context of what we call "the pole avoidance rule". The connection between the Fourier and Hilbert transforms is then used as an exercise in applying the developed methods.

(a) Fourier Transform Notations

Recall from Section 1 the statement of the Fourier transform, derived in Section 2,

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

We used lower case for a function of time like x(t), and upper case for the spectral components X(ω). Although convenient in many situations, this notation is a bit limiting for more general use, so we replace X(ω) with x^(ω). The above can then be written as,

$$\hat{x}(\omega) = k \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform}$$

$$x(t) = (1/2\pi k) \int_{-\infty}^{\infty} d\omega\, \hat{x}(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform}$$

Here we have added an arbitrary constant k to allow for other scalings of the Fourier Transform. The projection has k, the inversion has k^{-1} as shown. For us, k = 1, but other sources might have $k = (2\pi)^{-1}$ or perhaps $k = 1/\sqrt{2\pi}$ to make the two equations symmetric.

The first line defines the Fourier transform of some arbitrary function x(t), but the second line acts only on a function x^(ω) which is already a Fourier transform. We would like to have the second line act on an

arbitrary function as well. To do this, we replace x^(ω) by f(ω) and treat the second line as an operation one applies to some arbitrary function f(ω),

$$f^{\wedge -1}(t) = (1/2\pi k) \int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{+i\omega t} \qquad \text{inverse transform}$$

This then defines an operation performed on f(ω) to generate $f^{\wedge -1}(t)$ which is, by definition, the *inverse* Fourier transform of f(ω). So changing the dummy integration variable names both to u we get

$$f^{\wedge}(\omega) = k \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} \qquad \text{Fourier transform of f(u)}$$

$$f^{\wedge -1}(t) = (1/2\pi k) \int_{-\infty}^{\infty} du\, f(u)\, e^{+iut} \qquad \text{inverse Fourier transform of f(u)} \qquad \text{(C.1)}$$

We think of both these equations as defining certain operations on an arbitrary function f(u). The function f(u) must be in the class of functions described in Section 1(b) in order that the integral converge to a function, though the class rules may be violated if one allows the transform and/or its inverse to be a distribution.

From the first line of (C.1) we find that

$$f^{\wedge}(-\omega) = k \int_{-\infty}^{\infty} du\, f(u)\, e^{+i\omega u} = k \int_{-\infty}^{\infty} du\, f(-u)\, e^{-i\omega u} = [f(-u)]\hat{}(\omega)$$

from which we obtain

Fact 0: $[f(-u)]^{\wedge}(-\omega) = f^{\wedge}(\omega)$ (C.2)

From the second line of (C.1) we find

$$f^{\wedge -1}(-\omega) = (1/2\pi k) \int_{-\infty}^{\infty} du\, f(u)\, e^{-iu\omega} = (1/2\pi k^2)\; k \int_{-\infty}^{\infty} du\, f(u)\, e^{-iu\omega} = (1/2\pi k^2)\, f^{\wedge}(\omega) \quad .$$

We are dealing here with three distinct functions: f(u), f^(u) and $f^{\wedge -1}(u)$, but we just showed that

Fact 1: $f^{\wedge -1}(u) = (1/2\pi k^2)\, f^{\wedge}(-u)$ (C.3)

so two of these three functions have a simple relationship.

Notice in Fact 1 that one cannot simply suppress the argument u on both sides since u appears on the left and –u appears on the right. When an argument is the same on both sides of an equation, or when an argument is not needed, it can be suppressed to reduce clutter.

The fact that the Fourier transform is valid means that the inverse Fourier transform of the Fourier transform of a function *is* that function. Similarly, the Fourier transform of the inverse Fourier transform of a function *is* that function. This was clear in (1.1) and (1.2) and in the new notation this becomes

Fact 2: $[f\hat{}(\omega)]\hat{}^{-1}(t) = [f\hat{}^{-1}(\omega)]\hat{}(t) = f(t)$

or

$$[f\hat{}]\hat{}^{-1} = [f\hat{}^{-1}]\hat{} = f \qquad \text{(C.4)}$$

This provides an example of suppressing arguments to declutter a simple equation. Notice that the constant k does not appear in Fact 2. Consider then this statement of Fact 2,

$$(f\hat{})\hat{}^{-1}(t) = f(t)$$

Fact 1 applied to $f \rightarrow f\hat{}$ says

$$(f\hat{})\hat{}^{-1}(t) = (1/2\pi k^2)\,(f\hat{})\hat{}(-t)$$

so that

$$(1/2\pi k^2)\,(f\hat{})\hat{}(-t) = f(t)$$

or

$$(f\hat{})\hat{}(t) = 2\pi k^2\, f(-t)$$

which then gives

Fact 3: $$f\hat{}\hat{}(t) = 2\pi k^2\, f(-t) \qquad \text{(C.5)}$$

$$\text{FT of } f(t) = f\hat{}(t) \quad \Leftrightarrow \quad \text{FT of } f\hat{}(t) = 2\pi k^2 f(-t)$$

The first line says that applying the Fourier transform twice to a function gives $2\pi k^2$ times the function of negated argument (see Stakgold Vol II (5.55) with k = 1). Nothing new is happening here, it is all just notation. When $k = 1/\sqrt{2\pi}$ the factor $2\pi k^2 = 1$ on the second line which is a strong motivation for that scaling, but we had other motivations for k = 1 as outlined in Section 5.

The three Facts just stated are independent of the sign of the phase in the Fourier transform definition.

Operator Notation. An alternative notation similar to that used for Laplace transforms is the following:

$$f\hat{}(\omega) = \mathcal{F}[f(u),\omega] = \mathcal{F}[f,\omega] = \mathcal{F}\, f(\omega)$$

or

$$f\hat{} = \mathcal{F}\, f \qquad \text{(C.6)}$$

and for the inverse Fourier transform,

$$f\hat{}^{-1}(t) = \mathcal{F}^{-1}[f(u),t] = \mathcal{F}^{-1}[f,t] = \mathcal{F}^{-1} f(t)$$

or

$$f\hat{}^{-1} = \mathcal{F}^{-1} f \ . \qquad \text{(C.7)}$$

The idea here is that $\mathcal{F}$ f = g is a new function obtained by acting upon function f with the Fourier transform operator $\mathcal{F}$. Similarly $\mathcal{F}^{-1}$f = h is a new function obtained by acting upon function f with the inverse Fourier transform operator $\mathcal{F}^{-1}$. The three facts stated above can then be translated into this new notation.

Fact 1: $f\hat{}^{-1}(u) = (1/2\pi k^2)\, f\hat{}(-u) \quad \rightarrow \quad \mathcal{F}^{-1}[f(s),u] = (1/2\pi k^2)\, \mathcal{F}[f(s),-u]$

$\mathcal{F}^{-1}[f,u] = (1/2\pi k^2)\, \mathcal{F}[f,-u]$

$\mathcal{F}^{-1}f(u) = (1/2\pi k^2)\, \mathcal{F}\, f(-u)$ *(C.3)*

Fact 2: $[f\hat{}(\omega)]\hat{}^{-1}(t) = [f\hat{}^{-1}(\omega)]\hat{}(t) = f(t) \quad \rightarrow \quad \mathcal{F}^{-1}[\mathcal{F}[f(\omega),s],t] = \mathcal{F}[\mathcal{F}^{-1}[f(\omega),s],t] = f(t)$

$[f\hat{}]\hat{}^{-1} = [f\hat{}^{-1}]\hat{} = f \quad \rightarrow \quad \mathcal{F}^{-1}\mathcal{F}\, f = \mathcal{F}\mathcal{F}^{-1}f = f$ *(C.4)*

Fact 3: $f\hat{}\hat{}(t) = 2\pi k\, f(-t) \quad \rightarrow \quad \mathcal{F}[\mathcal{F}[f(\omega),s],t] = 2\pi k^2 f(-t)$

$\mathcal{F}[\mathcal{F}[f,s],t] = 2\pi k^2 f(-t)$

$\mathcal{F}^2 f(t) = 2\pi k^2 f(-t)$ *(C.5)*

The last line of Fact 2 has a particular appeal, since $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathbf{1}$, the identity operator.

(b) Principal Value Integrals and the Tick Notation

Consider the following integral,

$$\int_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x)\ .$$

If f(x) is non-vanishing at x = a, and if a is real, this integral runs right through a pole at x = a. If a were *complex*, then the integral would run above or below the pole and one would be less concerned. If the intention really is to run the integration right through the pole, it is useful to make that fact very clear using some kind of notation. One defines the notion of "going through the pole" as the following limiting operation,

$$\int_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x) \ = \ \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx\right]\ \frac{1}{x-a}\ f(x) \ \equiv \ \fint_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x)\ . \qquad \text{(C.8)}$$

Such an integration is referred to as a **Cauchy Principal Value** (or Principal Part) Integral. We defer to section (d) below the pf notation which formalizes the above definition as

$$\int_{-\infty}^{\infty} dx\ \mathrm{pf}\left(\frac{1}{x-a}\right) f(x) \ \equiv \ \fint_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x) \ \equiv \ \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx\right]\ \frac{1}{x-a}\ f(x)$$

As an example, consider this case where a = 0 and f(x) = 1,

$$\int_{-2}^{2} dx\ \frac{1}{x} \ = \ 0\ .$$

Although 1/x "blows up" at x = 0, this integral as defined above is exactly 0. All contributions to this integral are real, since 1/x is real, so the integral has no imaginary part. A simple argument for result zero is that the integration range is even while the integrand is odd under x → -x, so contributions from the left side of x=0 exactly cancel those from the right side of x=0. If the integrand contained some f(x) which was even under x → -x, the same argument would apply and the integral would be 0, but for general f(x) the integral would not be 0.

Comment: If one tries to evaluate the integral using ln(x), one gets a pre-limit result $\ln[\frac{a-\varepsilon}{a+\varepsilon}] + \ln[\frac{2}{-2}]$. After the limit, the first term gives ln[1] = 0 while the second term seems to give ln[-1] which one thinks of as ±iπ and we get a contradictory result that the real integral of a real integrand has an imaginary part. The problem is that this is a singular integral and the normal rules do not apply. The ±iπ reflects the fact that the integral is trying to avoid the pole by going above it or below it, whereas we really want to go right through it. Stakgold Vol. I Exercise 1.23 shows how this contradiction is resolved using a redefined log function, and the subject reappears below in our Pole Avoidance Rule discussion.

Here then are two notations used for integrals intended to run through poles,

$$\oint_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x) \ \equiv\ \text{P.V.}\ \int_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x) \ \equiv\ \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx\right]\ \frac{1}{x-a}\ f(x)\ . \qquad \text{(C.9)}$$

The tick mark on the integration symbol suggests the idea of running through the pole as the ε limit from the two sides, but is not easy to typeset, so one often sees the letters P.V, PV, p.v., v.p. , P or some other set of letters to indicate the principle part integration. We shall use the tick mark notation. Our example is then

$$\oint_{-2}^{2} dx\ \frac{1}{x} = 0\ . \qquad \text{(C.10)}$$

In the next several sections we shall examine some closely related examples of Fourier transforms, some of which require use of the Principal Value integral. The examples are later summarized in section (g).

(c) Example: f(u) = 1/u

Projection/Transform:

Let f(u) = 1/u. Using the regular Fourier transform requires that the integral go right through the pole, so we have

$$(1/u)\hat{}(\omega) \ =\ k\int_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} \ = k\ \oint_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} \quad . \qquad \text{(C.11)}$$

For ω ≠ 0 we can evaluate the integral this way,

$$\oint_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} \ =\ \oint_{-\infty}^{\infty} du(1/u)\,[-i\sin(\omega u)] = (-i)\ 2\int_{0}^{\infty} du\ \sin(\omega u)/u$$

$$= (-i)\, 2\, \text{sign}(\omega)\, \{ \int_0^\infty du\, \sin(|\omega|u)/u \} \quad = (-i)\, 2\, \text{sign}(\omega)\, \{ \int_0^\infty dx\, \text{sinc}(x) \} \qquad // \; x = |\omega|u$$

$$= (-i)\, 2\, \text{sign}(\omega)\ \{ \pi/2 \}$$

$$= -i\pi\, \text{sign}(\omega)\ . \qquad (C.12)$$

In the first step cos(ωu) was discarded since (1/u) cos(ωu) is an odd function of u. Once that is done, since sinc(x) = 1 at x = 0, there is no longer a pole at u = 0 so we have just a regular integral. The residual integral is half of (10.3).

If ω = 0, the integral is just $⨍_{-\infty}^{\infty} du(1/u) = 0$ based on the discussion of the previous section. One can combine these results by writing

$$⨍_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} \;=\; -i\pi\, \text{sgn}(\omega) \qquad (C.13)$$

where

$$\text{sgn}(\omega) \equiv \begin{cases} +1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases} \qquad // \text{ sometimes called signum}(\omega) \qquad (C.14)$$

This sgn(ω) function is related to the Heaviside step function by

$$\text{sgn}(\omega) = 2\theta(\omega) - 1 \qquad (C.15)$$

where in particular sgn(0) = 2θ(0) – 1 = 2(1/2)-1 = 0.

Our conclusion is that the Fourier transform of 1/u is given by

$$(1/u)\hat{}(\omega) = -i\pi k\, \text{sgn}(\omega)\ . \qquad (C.16)$$

If we treat sgn(ω) like any other function, we can suppress the ω argument to write

$$(1/u)\hat{} = -i\pi k\, \text{sgn}\quad . \qquad (C.17)$$

In the general case of f^(u) → f^ we could suppress the u, but once the function is stated (such as 1/u), it is difficult to suppress the u and still know what the function is. Note that the u in 1/u is just a dummy variable and we could just as well write

$$(1/t)\hat{}(\omega) = -i\pi k\, \text{sgn}(\omega)$$
$$(1/t)\hat{} = -i\pi k\, \text{sgn}\ . \qquad (C.18)$$

Inversion/Recovery:

How does the recovery work?

$$(1/2\pi k)\ \int_{-\infty}^{\infty} d\omega\ (1/t)\hat{}(\omega)\ e^{+i\omega t} = (1/2\pi k)\ \int_{-\infty}^{\infty} d\omega\ [-i\pi k\,\mathrm{sgn}(\omega)]\ e^{+i\omega t}$$

$$= (1/2\pi)(-i\pi)\ \int_{-\infty}^{\infty} d\omega\ \mathrm{sgn}(\omega)\ e^{+i\omega t} = (-i/2)\ \int_{-\infty}^{\infty} d\omega\ \mathrm{sgn}(\omega)\ [i \sin(\omega t]$$

$$= (-i/2)\ i\ 2\ \int_{0}^{\infty} d\omega\ \sin(\omega t) = \int_{0}^{\infty} d\omega\ \sin(\omega t) = -(1/t)\cos(\omega t)|_{0}^{\infty} = -(1/t)\ [\cos(\infty t) - \cos(0)]$$

$$= (1/t) \qquad \text{(C.19)}$$

The distributional trick is to set $\cos(\infty t) = 0$. In more detail,

$$\int_{0}^{\infty} d\omega\ \sin(\omega t) = \lim_{\varepsilon \to 0}\ [\ \int_{0}^{\infty} d\omega\ \sin(\omega t)\ e^{-\varepsilon\omega}\ ] = \lim_{\varepsilon \to 0} \frac{t}{t^2+\varepsilon^2} = \frac{1}{t} \quad . \qquad \text{(C.20)}$$

<u>Interchange:</u>

We have just shown that

$$(1/u)\hat{}(\omega) = -i\pi k\ \mathrm{sgn}(\omega)\ .$$

We can Fourier transform both sides to get

$$[(1/u)\hat{}(\omega)]\hat{}(t) = -i\pi k\ [\mathrm{sgn}(\omega)]\hat{}(t)$$

so that

$$[\mathrm{sgn}(\omega)]\hat{}(t) = (1/-i\pi k)\ \ [(1/u)\hat{}(\omega)]\hat{}(t)\ .$$

But according to Fact 3,

$$[(1/u)\hat{}(\omega)]\hat{}(t) = [(1/u)\hat{}]\hat{}(t) = (1/u)\hat{}\hat{}(t) = 2\pi k^2\ (1/-t) = -2\pi k^2/t\ .$$

Therefore

$$[\mathrm{sgn}(\omega)]\hat{}(t) = (1/-i\pi k)\ (-2\pi k^2/t) = 2k/(it) = -2ik(1/t)\ . \qquad \text{(C.21)}$$

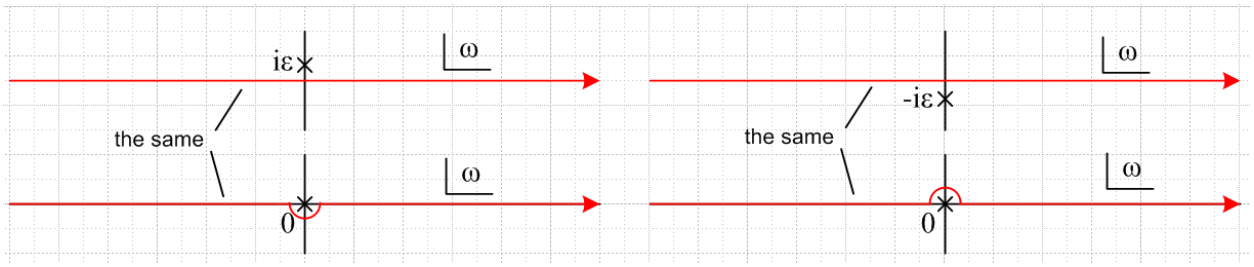
Thus we have learned the Fourier transform of the function sgn(ω). We could have computed this directly as follows:

$$[\mathrm{sgn}(\omega)]\hat{}(t) = k\ \int_{-\infty}^{\infty} du\ \mathrm{sgn}(u)\ e^{-itu} = k\ \int_{-\infty}^{\infty} du\ \mathrm{sgn}(u)\ [-i \sin(tu)] = (-i)\ 2k\ \int_{0}^{\infty} du\ \sin(tu)$$

$$= (-2ik)(-1/t)\cos(tu)|_{0}^{\infty} = (2ik/t)(0 - 1) = -2ik(1/t)\ .$$

(d) The Pole Avoidance Rule of Complex Integration

The upper picture on the left shows a real-axis integration contour in the ω-plane which passes just *below* a pole located at ω = +iε. We are interested in what happens as ε → 0. The upper picture on the right shows the contour passing just *above* a pole at ω = -iε and we have a similar interest there as ε → 0. The lower contour pictures show a certain contour deformation, and the pair of equations under each pair of drawings will be discussed below.



$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{1}{\omega - i\varepsilon} = \oint_{-\infty}^{\infty} d\omega\, f(\omega)\,(1/\omega) + i\pi\, f(0) \qquad \int_{-\infty}^{\infty} d\omega\, f(\omega)\frac{1}{\omega + i\varepsilon} = \oint_{-\infty}^{\infty} d\omega\, f(\omega)(1/\omega) - i\pi\, f(0)$$

$$\frac{1}{\omega - i\varepsilon} = \mathrm{pf}(1/\omega) + i\pi\delta(\omega) \qquad \frac{1}{\omega + i\varepsilon} = \mathrm{pf}(1/\omega) - i\pi\delta(\omega)$$

Fig C.1

We assume that f(ω) is such that the integrals converge and f(ω) is well defined at ω = 0. We are interested only in the limit ε→0.

Left Side. Consider the top left red arrow contour. If we try to take ε→0, the pole moves down and hits the contour, which is a poorly defined concept in complex integration. To prevent this from happening, we first make a tiny semi-circular deformation of the contour so it goes around ω = 0. One is certainly allowed to deform a contour and not change an integral, as long as the deformation hits no singularities. After doing this "for free" deformation, we *then* let the ε→0 so the pole moves down to the real axis. If we now evaluate the integral, we get two famous pieces: The first piece is the principle value integral discussed in section (b) above, namely,

$$\oint_{-\infty}^{\infty} d\omega\, f(\omega)\,(1/\omega) \equiv \lim_{\alpha\to 0}\left[\int_{-\infty}^{-\alpha} + \int_{\alpha}^{\infty}\right] d\omega\, f(\omega)\,(1/\omega)\,.$$

The second piece is a half-circle counterclockwise contour around the pole which gives *one half* the pole residue which result is then (1/2) 2πi f(0) = iπf(0). In general, if a contour goes some percentage around a pole, it picks up that percentage of the reside, which we now demonstrate, letting $\omega = Re^{i\theta}$,

$$\oint \frac{d\omega}{\omega} = \oint \frac{Re^{i\theta} i d\theta}{Re^{i\theta}} = i\int_{\theta_1}^{\theta_2} d\theta = i(\theta_2 - \theta_1)\,.$$

If we go half way around the pole, then $i(\theta_2-\theta_1) = i\pi$. So we have now derived the top left equation in Fig C.1.

Recall now from Appendix A that a distributional equation is one which gains its meaning when placed inside an integral. So consider

$$\int_{-\infty}^{\infty} d\omega\, f(\omega) \frac{1}{\omega - i\varepsilon} = \int_{-\infty}^{\infty} d\omega\, f(\omega) [\ pf(1/\omega) + i\pi\delta(\omega)\ ] = \int_{-\infty}^{\infty} d\omega\, f(\omega)\, pf(1/\omega) + i\pi\, f(0)\ .$$

Here both $pf(1/\omega)$ and $\delta(\omega)$ are symbolic functions as discussed in Appendix A. The meaning of $\delta(\omega)$ seems clear (sifting property) while the meaning of $pf(1/\omega)$ is precisely this:

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, pf(1/\omega) \equiv \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega)\ .$$

The letters pf stand for pseudofunction. Officially $f(\omega)$ should be a distribution theory "test function", but we just take it to be any reasonable function as described above. So we have now derived both equations on the left of Fig C.1.

Right Side. This is the same idea, but the required contour deformation is different, and since the semicircle then goes clockwise around the pole, we pick up minus half the residue which is $-i\pi f(0)$. Now both equations on the right are derived. We have then proven the following distributional equation:

The Pole Avoidance Rule:

$$\lim_{\varepsilon\to 0} \frac{1}{\omega \mp i\varepsilon} = pf(1/\omega) \pm i\pi\delta(\omega) \qquad \text{(C.22)}$$

Notice that this rule has no connection with phase sign conventions of the Fourier transform. It really has nothing at all to do with the Fourier transform in fact. This "rule" seems to have no official name, so we have made one up. For more discussion of this subject see Stakgold Vol. I page 50 (1.27) and previous pages.

(e) Example: f(u) = 1/(u±iε)

Here we always imply the limit $\varepsilon\to 0$.

Projection/Transform:

Using the Pole Avoidance Rule (C.21) above, we may compute the Fourier transform of this f(u) as follows:

$$\left(\frac{1}{u\pm i\varepsilon}\right)\hat{}(\omega) = k \int_{-\infty}^{\infty} du \frac{1}{u\pm i\varepsilon}\, e^{-i\omega u} = k \fint_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} \mp i\pi k$$

The principal value integral was found in (C.13) to be $-i\pi\, sgn(\omega)$, so we find that

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = -i\pi k\ \mathrm{sgn}(\omega) \mp i\pi k = -i\pi k\ (\mathrm{sgn}(\omega) \pm 1)\ .$$

Assume first the upper signs,

$$(\frac{1}{u+i\varepsilon})\hat{}(\omega) = -i\pi k\ \mathrm{sgn}(\omega) - i\pi k = -i\pi k\ (\mathrm{sgn}(\omega) + 1)$$

If $\omega > 0$, the result is $-2\pi ik$, and if $\omega < 0$ the result is 0. So

$$(\frac{1}{u+i\varepsilon})\hat{}(\omega) = -2\pi ik\theta(\omega)\ .$$

Now assume the lower signs

$$(\frac{1}{u-i\varepsilon})\hat{}(\omega) = -i\pi k\ \mathrm{sgn}(\omega) + i\pi k = -i\pi k\ (\mathrm{sgn}(\omega) - 1)$$

If $\omega > 0$, the result is 0. If $\omega < 0$, the result is $+2\pi ik$. So

$$(\frac{1}{u-i\varepsilon})\hat{}(\omega) = 2\pi ik\theta(-\omega)\ .$$

Combining these results we get the Fourier transform of $\frac{1}{u\pm i\varepsilon}$:

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = \mp\ 2\pi ik\ \theta(\pm\omega)\quad . \tag{C.23}$$

<u>Inversion/Recovery:</u>

$$(1/2\pi k)\ \int_{-\infty}^{\infty} d\omega\ (\frac{1}{u\pm i\varepsilon})\hat{}(\omega)\ e^{+i\omega t} = (1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ [\mp 2\pi i\ \theta(\pm\omega)]\ e^{+i\omega t}$$

$$= (1/2\pi)(\mp 2\pi i)\ \int_{-\infty}^{\infty} d\omega\ \theta(\pm\omega)\ e^{+i\omega t} = (\mp\ i)\ \int_{-\infty}^{\infty} d\omega\ \theta(\pm\omega)\ e^{+i\omega t}$$

First take the upper sign

$$= (-\ i)\ \int_{0}^{\infty} d\omega\ e^{+i\omega t}\ .$$

If t has a small positive imaginary part, the integral converges to (-1/it) to give

$$= (-\ i)\ (-1/it) = 1/t$$

but we write t as t+iε to show that it has this small positive imaginary part, so the result is

$$(1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ (\frac{1}{u+i\varepsilon})\hat{}(\omega)\ e^{+i\omega t} = \frac{1}{t+i\varepsilon}$$

as desired. Now we look at the lower sign

$$(+i)\int_{-\infty}^{0} d\omega\ e^{+i\omega t} = (+i)\ \int_{0}^{\infty}\ d\omega\ e^{-i\omega t}$$

If t has a small negative imaginary part, the integral converges to (+1/it) to give

$$= (+i)\ (+1/it)\ = 1/t\ = \frac{1}{t-i\varepsilon}$$

which is again the desired result.

Interchange:

We have just shown that

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega)\ = \mp\ 2\pi ik\ \theta(\pm\omega)$$

where it is understood that $\varepsilon \rightarrow 0$ on the left side, and either set of signs is valid.

We can Fourier transform both sides to get

$$[(\frac{1}{u\pm i\varepsilon})\hat{}(\omega)]\hat{}(t)\ = \mp\ 2\pi ik\ [\theta(\pm\omega)]\hat{}(t)\ .$$

But according to Fact 3,

$$[(\frac{1}{u\pm i\varepsilon})\hat{}(\omega)]\hat{}(t)\ \ = [(\frac{1}{u\pm i\varepsilon})\hat{}]\hat{}(t)\ = (\frac{1}{u\pm i\varepsilon})\hat{}\hat{}(t) = 2\pi k^2\ \frac{1}{-t\pm i\varepsilon}\ .$$

Therefore

$$[\theta(\pm\omega)]\hat{}(t)\ = (\mp\ 2\pi ik)^{-1}\ 2\pi k^2\ \frac{1}{-t\pm i\varepsilon} = \frac{k}{\mp i}\ \ \frac{1}{-t\pm i\varepsilon}\ \ = \frac{k}{\pm it+\varepsilon}$$

$$= \pm ik\ \frac{1}{-t\pm i\varepsilon}\ = \pm ik\ [\ pf(-1/t)\ \mp\ i\pi\delta(-t)\ ]\ = \pm ik\ [\ -pf(1/t)\ \mp\ i\pi\delta(t)\ ]\ \ =\ \mp\ i\ k\ pf(1/t) + \pi\ k\ \delta(t)$$

and so we learn the Fourier transform of the Heaviside step function with either sign argument. The result with the + sign is verified immediately below. A more standard naming of the arguments yields

$$[\theta(\pm t)]\hat{}(\omega) = \pm i\, k \frac{1}{-\omega \pm i\varepsilon} = \mp\, i\, k\, pf(1/\omega) + \pi\, k\, \delta(\omega) = k\, pf(\frac{1}{\pm i\omega}) + \pi\, k\, \delta(\omega) \quad . \tag{C.24}$$

(f) Example: f(u) = θ(u) using the Generalized Fourier Transform

We include this example only because it is related to the previous examples.

Projection/Transform:

From the first line of (C.1) the projection (Fourier transform) is given by

$$[\theta(u)]\hat{}(\omega) = k \int_{-\infty}^{\infty} du\, \theta(u)\, e^{-i\omega u} = k \int_{0}^{\infty} du\, e^{-i\omega u} \quad .$$

The generalized Fourier transform was defined in Section 6. Recall that the recovery contour in the inversion formula passes below all singularities of f(u), so for this example that contour runs just below the real axis so as to put the pole at u = 0 above the contour. Thus, we are really interested in the projection evaluated at ω-iε, so we then have a convergent integral,

$$[\theta(u)]\hat{}(\omega - i\varepsilon) = k \int_{0}^{\infty} du\, e^{-i(\omega - i\varepsilon)u} = k \int_{0}^{\infty} du\, e^{-(i\omega + \varepsilon)u} = k \frac{1}{i\omega + \varepsilon} .$$

Taking the limit ε→0 we obtain the generalized Fourier transform of θ(u) valid for general complex ω,

$$[\theta(u)]\hat{}(\omega) = \frac{k}{i\omega} \tag{C.25}$$

Inversion/Recovery:

We now evaluate the generalized Fourier inversion formula using the above projection,

$$(1/2\pi k) \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} d\omega\, f(\omega)\, e^{+i\omega t} = (1/2\pi k) \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} d\omega \frac{k}{i\omega} e^{+i\omega t}$$

$$= (1/2\pi i) \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} d\omega \frac{1}{\omega} e^{+i\omega t} = \theta(t) \qquad \text{as explained below :}$$

For t < 0, we close the contour down and pick up nothing giving 0.
For t > 0, we close the contour up and pick up the full pole residue to get (1/2πi)2πi= 1.

(g) Summary of Examples

Our convention uses k = 1, but see (C.1) for other conventions.

Function/Distribution	Fourier transform	Comment

x(t)	X(ω)	main text notation (k=1)
f(t)	$f^{\mathfrak{h}}(\omega)$	Appendix C notation
1/t	$-i\pi k\ \mathrm{sgn}(\omega)$	*(C.16)*
sgn(t)	$\frac{2k}{i\omega}$	*(C.21)*
$\lim_{\varepsilon\to 0}\frac{1}{t\pm i\varepsilon} = \mathrm{pf}(1/t) \mp i\pi\delta(t)$	$\mp\ 2\pi i k\ \theta(\pm\omega)$	*(C.23)*
$\theta(\pm t)$	$k\lim_{\varepsilon\to 0}\frac{1}{\pm i\omega+\varepsilon} = k\ \mathrm{pf}(\frac{1}{\pm i\omega}) + \pi k\ \delta(\omega)$	*(C.24)*
$\theta(t)$	$\frac{k}{i\omega}$ (generalized FT)	*(C.25)*

Comment: Looking back at (C.11) in light of the pf notation, the formally correct version of (C.11) would be this

$$[\mathrm{pf}(1/u)]\hat{}(\omega) = k\int_{-\infty}^{\infty} du\ \mathrm{pf}(1/u)\ e^{-i\omega u} = k\ \text{⨍}_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u} \qquad (C.11)$$

and then one would have this version of (C.18).

$$[\mathrm{pf}(1/t)]\hat{}(\omega) = -i\pi k\ \mathrm{sgn}(\omega)\ . \qquad (C.18)$$

Most tables of Fourier transforms omit the pf formality since things are clear without such notation.

(h) The Hilbert Transform and its relation to the Fourier Transform

The Hilbert Transform is defined as follows

$$f^{\mathfrak{h}}(t) \equiv (1/\pi)\ \text{⨍}_{-\infty}^{\infty} d\omega\ \frac{f(\omega)}{t-\omega} \equiv \mathcal{H}[\ f(x),t] = \mathcal{H}[f,t]\ . \qquad (C.26)$$

Again we show several different notations, though we shall mainly use $f^{\mathfrak{h}}(t)$. Changing the integration variable from ω to t' gives

$$f^{\mathfrak{h}}(t) = \text{⨍}_{-\infty}^{\infty} dt'\ \frac{(1/\pi)}{t-t'}\ f(t') = \int_{-\infty}^{\infty} dt'\ \mathrm{pf}(\frac{(1/\pi)}{t-t'})\ f(t') \qquad (C.27)$$

where we use the pf symbolic function introduced earlier. We recognize this as having the usual convolution form (3.1),

$$a(t) = \int_{-\infty}^{\infty} dt'\ b(t-t')c(t') \quad \text{sometimes written} \quad a = b * c \qquad (3.1)$$

where a = b * c becomes,

$$f^{h} = pf(\frac{1}{\pi t}) * f \quad . \tag{C.28}$$

The Convolution Theorem was stated in (3.6)

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega)\, C(\omega) \tag{3.6}$$

or in our new notation, where for example a^(ω) = kA(ω),

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad a\hat{}(\omega) = k^{-1}\, b\hat{}(\omega)\, c\hat{}(\omega) \ . \tag{3.6}$$

Therefore we obtain this diagonalized form of (C.27) in ω-space,

$$[f^{h}]\hat{}(\omega) = k^{-1}\, (\frac{1}{\pi t})\hat{}(\omega)\ \ f\hat{}(\omega) \tag{C.29}$$

But from (C.16) we know that (see Comment at the end of section g above)

$$(\frac{1}{\pi t})\hat{}(\omega) = -i\, k\, sgn(\omega) \tag{C.30}$$

Therefore (C.29) becomes (the scaling factor is now gone since both sides are Fourier transforms),

$$(f^{h})\hat{}(\omega) = -i\, sgn(\omega)\, f\hat{}(\omega) \quad . \tag{C.31}$$

This well-known result relates the Fourier transform of the Hilbert transform of a function directly to the Fourier transform of that function. We now make immediate use of this equation.

Since (C.31) can be applied to any function f (always assuming the Hilbert integral converges), we apply it first to g,

$$[(g)^{h}]\hat{}(\omega) = -i\, sgn(\omega)\, (g)\hat{}(\omega) \tag{C.32}$$

and then we set g = f^{h} to get

$$[(f^{h})^{h}]\hat{}(\omega) = -i\, sgn(\omega)\, (f^{h})\hat{}(\omega) \quad . \tag{C.33}$$

Installing (C.31) into the right side gives

$$[f^{hh}]\hat{}(\omega) = -i\, sgn(\omega)\, [\, -i\, sgn(\omega)\, f\hat{}(\omega)] = -\, f\hat{}(\omega). \tag{C.34}$$

Applying the inverse Fourier transform to both sides then gives

$$f^{hh}(t) = -f(t)$$

or

$$f^{hh} = -f \tag{C.35}$$

or in operator notation,

$$\mathcal{H}\,\mathcal{H}\,f = -f \quad . \tag{C.36}$$

This says that application of the Hilbert transform twice to a function gives that function preceded by a minus sign (compare to Fact 3). We can apply $\mathcal{H}^{-1}$ to both sides to get

$$\mathcal{H}^{-1}\,f \;= -\,\mathcal{H}\,f \tag{C.37}$$

and, in so doing, we have discovered the formula for the *inverse* Hilbert transform,

$$\mathcal{H}^{-1}f(t) \;=\; -\,(1/\pi)\,⨍_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{t-\omega} \;\equiv\; f^{h-1}(t) \quad . \tag{C.38}$$

We can then apply this last equation to f^h instead of f to get

$$\mathcal{H}^{-1}f^h(t) = \;-\,(1/\pi)\,⨍_{-\infty}^{\infty} d\omega\,\frac{f^h(\omega)}{t-\omega} \;= (f^h)^{h-1}(t) \;= f(t) \tag{C.39}$$

which gives us this Hilbert transform pair (in which the two members differ only by a sign),

$$\begin{aligned} f^h(t) &= \;\;(1/\pi)\,⨍_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{t-\omega} && \text{// projection = transform} \\ f(t) \;&= -\,(1/\pi)\,⨍_{-\infty}^{\infty} d\omega\,\frac{f^h(\omega)}{t-\omega} && \text{// inversion = recovery} \quad . \end{aligned} \tag{C.40}$$

We now make these replacements $\omega\to\omega'$, $f \to X$, $f^h \to X_h$, $t\to\omega$ to get

$$\begin{aligned} X_h(\omega) &= \;\;(1/\pi)\,⨍_{-\infty}^{\infty} d\omega'\,\frac{X(\omega')}{\omega-\omega'} && \text{// projection = transform} \\ X(\omega) \;&= -\,(1/\pi)\,⨍_{-\infty}^{\infty} d\omega'\,\frac{X_h(\omega')}{\omega-\omega'} && \text{// inversion = recovery} \end{aligned} \tag{C.41}$$

and this is a more familiar statement of the Hilbert transform pair. We have proved that this transform is valid by making use of its connection to the Fourier transform shown in (C.31).

About 13 pages of Hilbert transforms appear in Erdelyi ET2.

Example 1: Compute the Hilbert Transform of $f(\omega) = e^{i\beta\omega}$:

$$f^h(t) = \;-(1/\pi)\,⨍_{-\infty}^{\infty} d\omega\,\frac{1}{\omega-t}\;e^{i\beta\omega} \;=\; -(1/\pi)\,⨍_{-\infty}^{\infty} d\omega'\,\frac{1}{\omega'}\;e^{i\beta(\omega'+t)}$$

$$= -(1/\pi)\ e^{i\beta t} \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega'}\ e^{i\beta\omega'} = -(1/\pi)\ e^{i\beta t} \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega'}\ (i)\ \sin(\beta\omega') \quad // \ x = \beta\omega'$$

$$= -(i/\pi)\ e^{i\beta t}\ 2\ \mathrm{sgn}(\beta) \int_{0}^{\infty} dx\ \mathrm{sinc}(x) = -(i/\pi)\ e^{i\beta t}\ 2\ \mathrm{sgn}(\beta)\ \{\ \pi/2\ \} \qquad // \text{ as in (C.12)}$$

$$= -i\ \mathrm{sgn}(\beta)\ e^{i\beta t}$$

Therefore

$$f(\omega) = e^{i\beta\omega} \qquad \Leftrightarrow \qquad f^{h}(t) = -i\ \mathrm{sgn}(\beta) e^{i\beta t} \qquad \text{(C.42)}$$

or in the notation of the main text

$$X(\omega) = e^{i\beta\omega} \qquad \Leftrightarrow \qquad X_{h}(\omega) = -i\ \mathrm{sgn}(\beta) e^{i\beta\omega} \qquad \text{(C.43)}$$

Example 2: Equation (C.42) is valid for any β, so it is valid for -β. Since the Hilbert transform is linear we can then superpose exponentials to get the following correspondences,

$$f(\omega) = k \sum_{\beta} g_{\beta}\ e^{-i\beta\omega} \qquad \Leftrightarrow \qquad f^{h}(t) = -ik \sum_{\beta} \mathrm{sgn}(-\beta)\ g_{\beta}\ e^{-i\beta t}$$

$$f(\omega) = k \int_{-\infty}^{\infty} d\beta\ g(\beta)\ e^{-i\beta\omega} \qquad \Leftrightarrow \qquad f^{h}(t) = -ik \int_{-\infty}^{\infty} d\beta\ \mathrm{sgn}(-\beta)\ g(\beta)\ e^{-i\beta t}$$

The last line can be written

$$f(\omega) = \hat{g}(\omega) \qquad \Leftrightarrow \qquad f^{h}(t) = +i\ [\ \mathrm{sgn}(\beta) g(\beta)\ ]\hat{}(t)$$

which then says

$$[\hat{g}(\omega)]^{h}(t) = +i\ [\ \mathrm{sgn}(\beta) g(\beta)\ ]\hat{}(t)\ .$$

If we suppress ω, then change t to ω, then β to t, and replace function name g by f, this says

$$(\hat{f})^{h}(\omega) = +i\ [\ \mathrm{sgn}(t) f(t)\ ]\hat{}(\omega) \qquad \text{(C.44)}$$

which we can compare with (C.31) which has the transforms in the reverse order

$$(f^{h})\hat{}(\omega) = -i\ \mathrm{sgn}(\omega)\ \hat{f}(\omega) \quad . \qquad \textit{(C.31)}$$

Alternate derivation of (C.44).

Start with (C.26) applied to $\hat{f}$,

$$(f\hat{})^{h}(t) \equiv (1/\pi) \fint_{-\infty}^{\infty} d\omega \frac{f\hat{}(\omega)}{t-\omega}$$

or

$$(f\hat{})^{h} = pf(\frac{1}{\pi t}) * f\hat{}$$

The diagonalized version is then, using (C.30),

$$a\hat{}(\omega) = k^{-1}\, b\hat{}(\omega)\, c\hat{}(\omega)$$

or

$$((f\hat{})^{h})\hat{} = k^{-1}\,[-i\, k\, sgn(\omega)]\; f\hat{}\hat{}(\omega) = -i\, sgn(\omega)\, f\hat{}\hat{}(\omega)\ .$$

Use Fact 3 that $f\hat{}\hat{}(\omega) = 2\pi k^{2} f(-\omega)$ to get

$$((f\hat{})^{h})\hat{}(\omega) = -i\, sgn(\omega)\, 2\pi k^{2} f(-\omega) = 2\pi k^{2}\, i\, sgn(-\omega)\, f(-\omega)$$

Now take the Fourier transform of both sides

$$((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^{2} i\, [sgn(-s)\, f(-s)]\hat{}(\omega)$$

Use Fact 3 that $((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^{2}((f\hat{})^{h})(-\omega)$ to get

$$2\pi k^{2}((f\hat{})^{h})(-\omega) = 2\pi k^{2} i\, [sgn(-s)\, f(-s)]\hat{}(\omega)$$

or

$$((f\hat{})^{h})(-\omega) = i[sgn(-s)\, f(-s)]\hat{}(\omega)$$

or

$$((f\hat{})^{h})(\omega) = i\, [sgn(-s)\, f(-s)]\hat{}(-\omega)$$

Finally we get to use Fact 0 that $[f(-u)]\hat{}(-\omega) = f\hat{}(\omega)$ to obtain the final result

$$((f\hat{})^{h})(\omega) = i\, [sgn(s)\, f(s)]\hat{}(\omega)$$

which is (C.44).