Alternate derivation of (C.44) PhL 4.16.13

This is incorporated at the end of Appendix C. The task is to compute $(\hat{f})^{\mathbf{h}}$ as opposed to $(f^{\mathbf{h}})\hat{}$. This, this file is really no longer needed.

I want an expression for this object: $(\hat{f})^{\mathbf{h}}$ since I know it in the reverse order.

Start with (C.26) applied to $\hat{f}$

$$(\hat{f})^{\mathbf{h}}(t) \equiv (1/\pi) \fint_{-\infty}^{\infty} d\omega \frac{\hat{f}(\omega)}{t-\omega}$$

Rewrite as

$$(\hat{f})^{\mathbf{h}}(t) = \fint_{-\infty}^{\infty} dt' \frac{(1/\pi)}{t-t'} \hat{f}(t') = \int_{-\infty}^{\infty} dt' \; pf\left(\frac{(1/\pi)}{t-t'}\right) \hat{f}(t') \qquad (C.27)$$

or

$$(\hat{f})^{\mathbf{h}} = pf\left(\frac{1}{\pi t}\right) * \hat{f}$$

The diagonalized version is then

$$\hat{a}(\omega) = k^{-1} \hat{b}(\omega) \hat{c}(\omega)$$

or

$$((\hat{f})^{\mathbf{h}})\hat{} = k^{-1} pf\left(\frac{1}{\pi t}\right)\hat{}(\omega) \; \hat{\hat{f}}(\omega)$$

But use

$$\left(\frac{1}{\pi t}\right)\hat{}(\omega) = -i \, k \, sgn(\omega)$$

to get

$$((\hat{f})^{\mathbf{h}})\hat{} = k^{-1} [-i \, k \, sgn(\omega)] \; \hat{\hat{f}}(\omega) = -i \, sgn(\omega) \hat{\hat{f}}(\omega)$$

But we know that

$$\hat{\hat{f}}(\omega) = 2\pi k^2 f(-\omega) \qquad \text{from Fact 3}$$

We then have

$$((\hat{f})^{\mathbf{h}})\hat{}(\omega) = -i \, sgn(\omega) \, 2\pi k^2 f(-\omega) = -2\pi k^2 \, i \, sgn(\omega) f(-\omega) = 2\pi k^2 \, i \, sgn(-\omega) f(-\omega)$$

Now take the FT on both sides

$$((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^2 i\,[\mathrm{sgn}(-s)\, f(-s)]\hat{}(\omega)$$

Use Fact 3 on the LHS to get

$$((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^2((f\hat{})^{h})(-\omega)$$

Then we have

$$2\pi k^2((f\hat{})^{h})(-\omega) = 2\pi k^2 i\,[\mathrm{sgn}(-s)\, f(-s)]\hat{}(\omega)$$

$$((f\hat{})^{h})(-\omega) = i[\mathrm{sgn}(-s)\, f(-s)]\hat{}(\omega)$$

$$((f\hat{})^{h})(\omega) = i\,[\mathrm{sgn}(-s)\, f(-s)]\hat{}(-\omega) = i[\mathrm{sgn}(s)\, f(s)]\hat{}(\omega)$$