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(c) Pulse Trains with Repeated Amplitude Sequences

Suppose a pulse train has period $2T_1$ during which time the pulse

Consider a pulse train composed of some general pulse $x_{pulse}(t)$ whose amplitudes are repeated sequences of A,B. We let each pulse have width T_1 so the repeating sequence is $2T_1$ long. Then we think of these two pulses as forming a single longer pulse x_{Pulse} of length $2T_1$. Then

$$x_{Pulse}(t) = A\, x_{pulse}(t) \qquad 0 < t < T_1$$
$$x_{Pulse}(t) = B\, x_{pulse}(t-T_1) \quad T_1 < t < 2T_1$$

We then apply our Fourier Series results of box (15.12) using X_{Pulse} for the pulse and $2T_1$ for the period.

$$x(t) = \sum_{n=-\infty}^{\infty} x_{PULSE}(t - n2T_1)$$

$$C_m = (1/2T_1) \int_0^{2T_1} dt\, x_{PULSE}(t)\, e^{-im\omega_1 t/2}$$

$$= (1/2T_1)\, A \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2} + (1/2T_1)\, B \int_{T_1}^{2T_1} dt\, x_{pulse}(t-T_1)\, e^{-im\omega_1 t/2}$$

$$= (1/2T_1)\, A \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2} + (1/2T_1)\, B \int_0^{T_1} dt'\, x_{pulse}(t')\, e^{-im\omega_1 (t'+T1)/2}$$

$$= (1/2T_1)\, A \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2} + (1/2T_1)\, B e^{-im\omega_1 (T_1/2)} \int_0^{T_1} dt'\, x_{pulse}(t')\, e^{-im\omega_1 t'/2}$$

$$= (1/2T_1)\, [\, A + B\, e^{-im\omega_1 (T_1/2}\,] \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2}$$

$$= (1/2)\, [\, A + B\, e^{-im\omega_1 (T_1/2}\,](1/T_1) \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2}$$

$$= (1/2)\, [\, A + B\, e^{-im\omega_1 (T_1/2}\,] c_{m/2}$$

Thus we find that

$$C_m = (1/2)\ [\ A + B\ e^{-im\omega_1(T_1/2)}\ ]c_{m/2}$$

where the c_n go with $x_{pulse}(t)$ having period T_1. But

$$\omega_1 T_1/2 = \pi \text{ and then } e^{-im\omega_1(T_1/2)} = e^{-im\pi} = (-1)^m$$

So we then have

$$C_m = (1/2)\ [\ A + B(-1)^m\ ]c_{m/2} = (1/T_1)X_{pulse}(m\omega_1/2)\ (1/2)\ [\ A + B(-1)^m\ ]$$

Our spectrum is then for the $2T_1$ period pulse train is then

$$X(\omega) = \sum_{m=-\infty}^{\infty} C(\omega)\ 2\pi\ \delta(\omega - m\omega_1/2) = \sum_{m=-\infty}^{\infty} C_m\ 2\pi\ \delta(\omega - m\omega_1/2)$$

$$= (1/2)\ [\ A + B(-1)^m\ ]c_{m/2}\ 2\pi\ \delta(\omega - m\omega_1/2)$$

But of course

$$c_{m/2} = c(m\omega_1/2) = (1/T_1)X_{pulse}(m\omega_1/2)$$

so we seem then to get

$$X(\omega) = \sum_{m=-\infty}^{\infty} (1/2)\ (1/T_1)X_{pulse}(m\omega_1/2)\ [\ A + B(-1)^m\ ]\ 2\pi\ \delta(\omega - m\omega_1/2)$$

which compare to my Section 34 result

$$X(\omega) = X_{pulse}(\omega)\ (1/2)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ A + B(-1)^m\ ]\ \delta(\omega - m\omega_1/2) \qquad (34.18)$$

They agree!! So here is some support for my result.

Next, from (33.29) we get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |C_m|^2\ \delta(\omega - m\omega_1/2) =$$

$$\sum_{m=-\infty}^{\infty} |(1/T_1)X_{pulse}(m\omega_1/2)\ (1/2)\ [\ A + B(-1)^m\ ]\ |^2\ \delta(\omega - m\omega_1/2)$$

$$= (1/T_1)^2\,(1/4)\sum_{m=-\infty}^{\infty} |X_{pulse}(m\omega_1/2)|^2\,|[\,A + B(-1)^m\,]|^2\,\delta(\omega - m\omega_1/2)$$

$$= (1/T_1)^2\,|X_{pulse}(\omega)|^2\,(1/4)\sum_{m=-\infty}^{\infty} |[\,A + B(-1)^m\,]|^2\,\delta(\omega - m\omega_1/2)$$

$$= (1/T_1)\mathcal{P}_{pulse}(\omega)\,(1/4)\sum_{m=-\infty}^{\infty} |[\,A + B(-1)^m\,]|^2\,2\pi\,\delta(\omega - m\omega_1/2)$$

and this agrees as well.

We have a small problem here.

Our starting point is (34.8) with (34.7), where we assume N is large and later we will take $N\to\infty$:

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\,X'(\omega) = X_{pulse}(\omega)\,X''(z) \qquad (34.6)$$

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2\,(1/T_1)^2\,|X'(\omega)|^2 = |X_{pulse}(\omega)|^2\,|X''(z)|^2 \qquad z = e^{i\omega T_1} \qquad (34.8)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n\,e^{-i\omega n T_1}. \qquad (34.7)$$

The main problem is to compute X''(z) and then square it. We have

$$\sum_{n=-N}^{N} y_n\,e^{-i\omega n T_1} = A\sum_{n\ even} e^{-i\omega n T_1} + B\sum_{n\ odd} e^{-i\omega n T_1}$$

Now process the sums as follows, where

$$\sum_{n\ even} e^{-i\omega n T_1} = \sum_{m=-N/2}^{N/2} e^{-i\omega(2m)T_1} \quad \text{where we used } n = 2m$$

$$\sum_{n\ odd} e^{-i\omega n T_1} = \sum_{n=(-N-1)/2}^{(N-1)/2} e^{-i\omega(2m+1)T_1} \quad \text{where we used } n = 2m + 1$$

We assume N is very large, so we regard $(N\pm1)/2 \approx N/2$. We then find

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n e^{-i\omega n T_1} = [\, A + B\, e^{-i\omega T_1}] \sum_{m=-N/2}^{N/2} e^{-i\omega(2m)T_1}\,.$$

We now use (13.3),

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N)\,, \qquad -\infty < k < \infty \qquad (13.3)$$

to write

$$\sum_{m=-N/2}^{N/2} e^{-i\omega(2m)T_1} = 2\pi\, \delta_5(2\omega T_1, N/2)$$

where δ_5 and δ_6 to come are explained in Appendix A. Therefore

$$X''(z) = [\, A + B\, e^{-i\omega T_1}]\, 2\pi \delta_5(2\omega T_1, N/2) \qquad (34.16)$$

Using the Appendix A result,

$$\lim_{N\to\infty} \delta_5(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.19)$$

we obtain the $N\to\infty$ limit for our spectrum

$$X''(z) = [\, A + B\, e^{-i\omega T_1}]\, 2\pi \sum_{m=-\infty}^{\infty} \delta(2\omega T_1 - 2\pi m)$$

$$= (1/2)[\, A + B\, e^{-i\omega T_1}]\, (1/T_1)\; 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \qquad (34.17)$$

and correspondingly

$$X(\omega) = (1/2)\, X_{pulse}(\omega)\, [\, A + B\, e^{-i\omega T_1}]\, \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \qquad (34.18)$$

There is a certain logic to the $[\, A + B\, e^{-i\omega T_1}]$ factor. If we set A = K and B = 0 we get one result, and if we set A = 0 and B = K we get the same result multiplied by $e^{-i\omega n T_1}$. The second pulse train is just the first pulse train shifted T_1 units to the right, and this adds phase $e^{-i\omega n T_1}$ as in (12.1).

If we were to square (34.18)and use our usual $2\pi\delta(0) = 2N+1$ association, we get a result that is off by a factor of 2. The reason is that our pre-limit sums are going from -N/2 to N/2, so we would get the right answer if we were to adjust and say $2\pi\delta(0) = N+1$. Rather than make an arm-waving argument to this

effect, it is safer to continue along with our pre-limit expressions, having paused to take the limit for the spectrum X(ω) as in (34.18).

So, backing off again from limit, we square (34.16) to get

$$|X''(z)|^2 = |A + Be^{-i\omega T_1}|^2 \, [2\pi \, \delta_5(2\omega T_1, N/2)]^2$$

Then from (A.20) applied with N → N/2

$$\delta_6(k,N/2) \equiv \frac{1}{2\pi} \, \frac{[2\pi\delta_5(k,N/2)]^2}{(N+1)} \qquad \textit{(A.20)}$$

we get

$$|X''(z)|^2 = |A + Be^{-i\omega T_1}|^2 \, [2\pi \, \delta_5(2\omega T_1, N/2)]^2$$

or

$$\frac{|X''(z)|^2}{2\pi(N+1)} = |A + Be^{-i\omega T_1}|^2 \, \{ \frac{1}{2\pi} \, \frac{[2\pi\delta_5(2\omega T_1, N/2)]^2}{(N+1)} \} \; = |A + Be^{-i\omega T_1}|^2 \; \delta_6(2\omega T_1, N/2) \, .$$

Now for large N we ignore the difference between N and N + 1 and so on, so we divide both sides by 2 to get,

$$\frac{|X''(z)|^2}{2\pi(2N+1)} \; = \; (1/2) \, |A + Be^{-i\omega T_1}|^2 \; \delta_6(2\omega T_1, N/2)$$

Notice that a very important factor of 1/2 appears on the right in the last step. We now add back the squared pulse spectrum to get

$$\frac{|X(\omega)|^2}{2\pi(2N+1)} \; = \; |X_{\mathbf{pulse}}(\omega)|^2 \, \frac{|X''(z)|^2}{2\pi(2N+1)} \; = \; |X_{\mathbf{pulse}}(\omega)|^2 \; (1/2)|A + Be^{-i\omega T_1}|^2 \; \delta_6(2\omega T_1, N/2)$$

If we divide both sides by T_1 the left side is $\frac{|X(\omega)|^2}{2\pi T}$ where T is the length of the pulse train and this in turn equals $\mathcal{P}(\omega)$ all as shown in box (34.4). So for large N we have shown that

$$\mathcal{P}(\omega) \; = \; |X_{\mathbf{pulse}}(\omega)|^2 \, (1/T_1)(1/2) \, |A + Be^{-i\omega T_1}|^2 \; \delta_6(2\omega T_1, N/2)$$

$$= \; \mathcal{P}_{\mathbf{pulse}}(\omega)(1/2) \, |A + Be^{-i\omega T_1}|^2 \; 2\pi \, \delta_6(2\omega T_1, N/2) \qquad (34.19)$$

Now at last we take the limit N→∞ and use

$$\lim_{N\to\infty} \delta_6(k,N) \; = \; \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m) \qquad \textit{(A.21)}$$

to get our desired infinite pulse train result

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)(1/2)\,|A + Be^{-i\omega T_1}|^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(2\omega T_1 - 2\pi m)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)(1/4)\,|A + Be^{-i\omega T_1}|^2\,(1/T_1) \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - m\omega_1/2) \qquad (34.20)$$

Meanwhile,

$$|A + Be^{-i\omega T_1}|^2 = |A|^2 + |B|^2 + 2|A||B|\cos(\omega T_1)$$

so here is the final result:

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\,(1/4)(1/T_1)\,\{\,|A|^2 + |B|^2 + 2|A||B|\cos(\omega T_1)\} \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - m\omega_1/2) \qquad (34.21)$$

When $\cos(\omega T_1)$ is brought inside the sum, it becomes $\cos(T_1 m\omega_1/2) = \cos(m\pi) = (-1)^{m-1}$, so the result can also be written

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\,(1/4)(1/T_1) \sum_{m=-\infty}^{\infty} \{\,|A|^2 + |B|^2 - 2|A||B|\,(-1)^m\}\,2\pi\,\delta(\omega - m\omega_1/2) \qquad (34.22)$$

Special case 1: Suppose A = 1 and B = -1. Then

$$\{\,|A|^2 + |B|^2 - 2|A||B|\,(-1)^m\} = 2\{1 - (-1)^m\}$$

This vanishes for even m and equals 4 for odd m, so we then get

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\omega_1 \sum_{m = \pm \text{ odd}} 2\pi\,\delta(\omega - m\omega_1/2)$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{m = \pm \text{ odd}} 2\pi\,\delta(x - m/2) \qquad \text{where } x = \omega/\omega_1 \qquad (34.23)$$

Meanwhile, the spectrum from (34.18) becomes

$$X(\omega) = (1/2)\,X_{\mathbf{pulse}}(\omega)\,[\,1 - e^{-i\omega T_1}]\,\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

We can write

$$(1/2)\,[\,1 - e^{-i\omega T_1}] = i\,e^{-i\omega T_1/2}\sin(\omega T_1/2)$$

Since $\omega T_1/2 \rightarrow m\pi/2$ under that action of the delta function, we have $\sin(m\pi/2)$ which vanishes for even m and equals $(-1)^{m-1}$ for odd m. The phasor becomes $e^{-i\omega T_1/2} = e^{-im\pi/2} = (i)^{-m}$ so we get

$$X(\omega) = X_{pulse}(\omega) \sum_{m = \pm \text{ odd}} i\ (i)^{-m}(-1)^{m-1}\ \omega_1\ \delta(\omega - m\omega_1/2)$$

$$= X_{pulse}(\omega) \sum_{m = \pm \text{ odd}} (i)^{m-1}\ \omega_1\ \delta(\omega - m\omega_1/2)$$

$$= X_{pulse}(\omega) \sum_{m = \pm \text{ odd}} (-1)^{(m-1)/2}\ \omega_1\ \delta(\omega - m\omega_1/2) \qquad (34.24)$$

<u>Square wave pulse train with peak-to-peak = 2 units:</u>

Specializing further to the case of a square pulse, we can write from (9.2) and box (34.4)

$$X_{pulse}(\omega) = T_1\ \text{sinc}(\omega T_1/2) = T_1 \sin(\omega T_1/2)/(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (1/2\pi)\ T_1\ \text{sinc}^2(\omega T_1/2) = (1/\omega_1) \sin^2(\omega T_1/2)/(\omega T_1/2)^2$$

When these are evaluated at $\omega = m\omega_1/2$ we get $\omega T_1/2 = m\pi/2$ and we know $\sin(m\pi/2) = (-1)^{m-1}$, so

$$X_{pulse}(\omega) = T_1\ (-1)^{m-1}/(m\pi/2) = (2/\pi)\ T_1\ (-1)^{m-1}/m$$

$$\mathcal{P}_{pulse}(\omega) = (1/\omega_1)T_1 / (m\pi/2)^2 = (2/\pi)^2\ (1/\omega_1)T_1/m^2$$

Then we find for this square wave pulse train

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m = \pm \text{ odd}} 2\pi\ \delta(\omega - m\omega_1/2) =$$

$$= (2/\pi)^2\ \ T_1 \sum_{m = \pm \text{ odd}} (1/m^2)\ \ 2\pi\ \delta(\omega - m\omega_1/2)$$

$$X(\omega) = X_{pulse}(\omega) \sum_{m = \pm \text{ odd}} (-1)^{(m-1)/2}\ \omega_1\ \delta(\omega - m\omega_1/2)$$

$$= \sum_{m = \pm \text{ odd}} (-1)^{(m-1)/2}\ \ (2/\pi)\ T_1\ (-1)^{m-1}/m \qquad \omega_1\ \delta(\omega - m\omega_1/2)$$

$$= (2/\pi)\ T_1\ \omega_1 \sum_{m = \pm \text{ odd}} (-1)^{(m-1)/2}\ \ (-1)^{m-1}/m \qquad \delta(\omega - m\omega_1/2)$$

$$= (2/\pi)\, T_1\, \omega_1 \sum_{m = \pm \text{ odd}} (-1)^{(m-1)/2}\, (1/m) \quad \delta(\omega - m\omega_1/2)$$

We shall now challenge this result for various special cases.

Special Case 0. If A = 1 and B = -1 we get

$$\{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} = 2(1-\cos(\omega T_1)) = 4 \sin^2(\omega T_1/2)$$

so then

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\, (1/\omega_1) \sin^2(\omega T_1/2) \sum_{m = -\infty}^{\infty} \delta(\omega - m\omega_1/2) \tag{34.21}$$

and

If we convert this to dimensionless frequency $x = \omega/\omega_1$ it becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega) \sum_{m = -\infty}^{\infty} \sin^2(m\pi/2)\delta(x - [m/2]) \tag{34.21}$$

All even m terms vanish so this becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega) \sum_{m = \pm \text{odd}} \delta(x - [m/2]) \tag{34.21}$$

Special Case 1. $x_{\mathtt{pulse}}(t)$ is a box of width T_1 and height 1. From (9.2) we then have

$$X_{\mathtt{pulse}}(\omega) = T_1 \operatorname{sinc}(\omega T_1/2)$$

so that

$$\mathcal{P}(\omega) = (1/4) \operatorname{sinc}^2(\omega T_1/2)\, \{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} \sum_{m = -\infty}^{\infty} \delta(\omega - m\omega_1/2) \tag{34.22}$$

Special Case 1a: Let A = B = 1. First, let's calculate this another way. If the pulse train is a DC signal x(t) = 1, it must be that $\mathcal{P}(\omega) = k\delta(\omega)$ for some k. The total power is then $P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) = k$. But for a load of R = 1Ω, we know that P will be 1 watt for A = 1 volt, so k = 1 and $\mathcal{P}(\omega) = \delta(\omega)$.

So how exactly is this going to happen? First

$$\{ |A|^2 + |B|^2 + 2|A||B| \cos(\omega T_1)\} = 2[1+\cos(\omega T_1)] = 4 \cos^2(\omega T_1/2)$$

Evaluated at $\omega = m\omega_1/2$ we get $\omega T_1/2 = m\pi/2$. So then

$$\mathcal{P}(\omega) = (1/4) \sum_{m=-\infty}^{\infty} \text{sinc}^2(m\pi/2) \{4 \cos^2(m\pi/2)\} \delta(\omega - m\omega_1/2) = \delta(\omega)$$

For all integer $m \neq 0$ we have a factor $2\sin(m\pi/2)\cos(m\pi/2) = \sin(m\pi) = 0$ so there is no contribution. Then for m = 0, sinc = 1 and cos = 1 and the result is $\mathcal{P}(\omega) = \delta(\omega)$, as expected.

Special Case 1b: Let A = 1 and B = 0. Then

$$\mathcal{P}(\omega) = (1/4) \text{sinc}^2(\omega T_1/2) \{1\} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$= (1/4) \sum_{m=-\infty}^{\infty} \text{sinc}^2(m\pi/2) \delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m\neq 0} (m\pi/2)^{-2} \sin^2(m\pi/2) \delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m \text{ odd}} (m\pi/2)^{-2} \sin^2(m\pi/2) \delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) \sum_{m \text{ odd}} (m\pi/2)^{-2} \delta(\omega - m\omega_1/2)$$

$$= (1/4)\delta(\omega) + (1/4) (2/\pi)^2 \sum_{m \text{ odd}} \frac{1}{m^2} \delta(\omega - m\omega_1/2) \qquad (34.23)$$

For verification, we recall (33.27) that $\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2 \delta(\omega - m\omega_1)$ and we consider the waveform shown in Fig ** with coefficients as in (16.2a). For that waveform we get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1\tau/T_1)^2 \text{sinc}^2(m\pi\tau/T_1) \delta(\omega - m\omega_1)$$

To get a 50% duty cycle, we set $\tau = T_1/2$ to get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1/2)^2 \text{sinc}^2(m\pi/2) \delta(\omega - m\omega_1)$$

But to get our current A = 1 B = 0 waveform, we have to scale $T_1/2 \rightarrow T_1$ in Fig ** . This means we are taking $T_1 \rightarrow 2T_1$ so that means $\omega_1 \rightarrow \omega_1/2$ and then we get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} (1/2)^2 \text{sinc}^2(m\pi/2) \delta(\omega - m\omega_1/2)$$

which agrees an early line of our result quoted above. This can be written, by the way, as

$$\mathcal{P}(\omega) = (1/4)\,\delta(\omega) + \sum_{m=\pm odd} (1/2)^2\,\mathrm{sinc}^2(m\pi/2)\;\delta(\omega - m\omega_1/2)$$

The origin for t = 0 in the waveform plays no role in $\mathcal{P}(\omega)$ since the phase shift dies off in $|X(\omega)|^2$. The DC component is consistent with an average DC level of 1/2.

Special Case 1c: Let A = 1 and B = -1. Then

$$\mathcal{P}(\omega) = (1/4)\,\mathrm{sinc}^2(\omega T_1/2)\,\{\,|A|^2 + |B|^2 + 2|A||B|\cos(\omega T_1)\}\sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_1/2)$$

$$= (1/4)\,\mathrm{sinc}^2(\omega T_1/2)\,\{\,2 - 2\cos(\omega T_1)\}\sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_1/2)$$

$$= \mathrm{sinc}^2(\omega T_1/2)\,\sin^2(\omega T_1/2)\sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_1/2)$$

$$= \sum_{m=-\infty}^{\infty}\mathrm{sinc}^2(m\pi/2)\,\sin^2(m\pi/2)\;\delta(\omega - m\omega_1/2)$$

$$= \sum_{m=-\infty}^{\infty}(m\pi/2)^{-2}\,\sin^4(m\pi/2)\;\delta(\omega - m\omega_1/2)$$

$$= \sum_{m=\pm odd}(m\pi/2)^{-2}\,\sin^4(m\pi/2)\;\delta(\omega - m\omega_1/2) = \sum_{m=\pm odd}(m\pi/2)^{-2}\;\delta(\omega - m\omega_1/2)$$

so the final result is

$$\mathcal{P}(\omega) = (2/\pi)^2 \sum_{m=\pm odd}\frac{1}{m^2}\;\delta(\omega - m\omega_1/2) \qquad (34.24)$$

Verification: If we add a DC offset of +1 to our case 1c waveform, we will get

$$\mathcal{P}(\omega) = \delta(\omega) + (2/\pi)^2 \sum_{m=\pm odd}\frac{1}{m^2}\;\delta(\omega - m\omega_1/2)$$

But this is the same waveform as in Special Case 1b with A = 2 and B = 0, so $\mathcal{P}(\omega)$ should be 4x the result of Special Case 1b, which in fact it is.

Exercises for the Reader:

(a) If the repeating amplitude sequence is A,B,C then

$$\mathcal{P}(\omega) = |X_{\mathbf{pulse}}(\omega)|^2 \, (1/T_1)(1/3) \, |A + Be^{-i\omega T_1} + C\, e^{-i2\omega T_1}|^2 \sum_{m=-\infty}^{\infty} \delta(3\omega T_1 - 2\pi m) \tag{34.25}$$

(b) If the repeating sequence is $A_0, A_1 .... A_{M-1}$ then

$$\mathcal{P}(\omega) = |X_{\mathbf{pulse}}(\omega)|^2 \, (1/T_1)(1/M) \, \Big| \sum_{k=0}^{M-1} A_k e^{-ik\omega T_1} \Big|^2 \sum_{m=-\infty}^{\infty} \delta(M\omega T_1 - 2\pi m) \tag{34.26}$$