Pondering Group Delay PhL 3.4.13

Here I was trying to figure this out. The web just gives the formula for group delay but does not say why it has that name, nor what group is being delayed, etc. I had to roll my own as usual. It now appears as section 21(b) in spectral doc.

$U(\omega) = G(\omega)\, F(\omega)$

$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, G(\omega)\, F(\omega)\, e^{+i\omega t}$

Now suppose F(ω) is for some pulse,

$U(\omega) = G(\omega)\, X_{\mathbf{pulse}}(\omega)$

$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, G(\omega)\, X_{\mathbf{pulse}}(\omega)\, e^{+i\omega t}$

Suppose $G(\omega) = |G(\omega)|\, e^{-\varphi(\omega)}$

$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{\mathbf{pulse}}(\omega)\, e^{+i\omega t}$

Plan A. Let's try f(t) = δ(t) so that $X_{\mathbf{pulse}}(\omega) = 1$ from (8.2). Then

$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, e^{+i\omega t}$

Let's assume that most of the contribution comes from the region near ω = 0. Does this assumption have a name. That name is not "stationary phase" because that is the limit of an integral for some large x. I don't see how dφ/dω is going to "come in". It must be a parts integration.

Plan B. Web says it applies to a "narrow band" signal, so my δ is the wrong thing to work with which has a very wide band spectrum. So let's assume that $X_{\mathbf{pulse}}(\omega)$ has a narrow band around some ω_2. In the extreme case we could try $X_{\mathbf{pulse}}(\omega) = \delta(\omega-\omega_2)$ and then above we get

$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{\mathbf{pulse}}(\omega)\, e^{+i\omega t}$

$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, \delta(\omega-\omega_2)\, e^{+i\omega t}$

$= (1/2\pi)\, |G(\omega_2)|\, e^{-i\varphi(\omega 2)}\, e^{+i\omega 2 t}$

This is no good because Xpulse(t) = 1 and you have no spatial localization. I have no idea right now where that derivative thing is coming from! Hard to get info on this, so I should incorporate it when I learn how it works. I know I have done this before.

Plan C. Suppose we make the thing less narrow, so that it has a spectrum from ω_2-a to ω_2+ a. Then we get roughly

$$u(t) = (1/2\pi)\,|G(\omega_2)| \int_{\omega_2-a}^{\omega_2+a} d\omega\, e^{-i\varphi(\omega)}\, X_{pulse}(\omega)\, e^{+i\omega t}$$

Plan D. Suppose $x_{pulse}(t)$ has max at t = 0. Where does u(t) have its max

$$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{pulse}(\omega)\, e^{+i\omega t}$$

$$du/dt = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{pulse}(\omega)\, d_t e^{+i\omega t}$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{pulse}(\omega)\, i\omega\, e^{+i\omega t} = 0$$

That does nothing for me.

Every website just quotes the definition of group delay and gives no derivation or explanation!

Plan E. Wiki suggests using this function

$$x_{pulse}(t) = a(t)\cos(\omega_2 t+\theta)$$

Again they quote a result but don't derive it/

I will try this instead

$$x_{pulse}(t) = a(t)\exp(i\omega_2 t)$$

What is the spectrum of such a pulse"

$$X(\omega) = \int_{-\infty}^{\infty} dt\, a(t)\exp(i\omega_2 t)\, e^{-i\omega t}$$

$$\approx a(0) \int_{-\infty}^{\infty} dt\, xp(i\omega_2 t)\, e^{-i\omega t}$$

$$\approx a(0)\, \delta(\omega-\omega_2)$$

Put this in to get

$$u(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, X_{pulse}(\omega)\, e^{+i\omega t}$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |G(\omega)|\, e^{-i\varphi(\omega)}\, a(0)\, \delta(\omega-\omega_2)\, e^{+i\omega t}$$

$$\approx (1/2\pi)\, |G(\omega_2)|\, e^{-i\varphi(\omega 2)}\, a(0)\, e^{+i\omega 2 t}$$

Nothing is happening for me here! This question is hard to get answered!

Plan F. Let's try it for the simplest possible case treated in Lam page 9. He uses

$$G(\omega) = e^{-ik\omega/\omega c} \quad \text{in range} \quad (-\omega_c, \omega_c)$$

This is a narrow passband filter, it is G = 0 elsewhere!

Then I would get

$$u(t) = (1/2\pi) \int_{-\omega c}^{\omega c} d\omega\; e^{-ik\omega/\omega c}\, X_{pulse}(\omega)\, e^{+i\omega t}$$

We will use x(t) = δ(t) with X_{pulse}= 1 and talk "impulse response". Then we have

$$u(t) = (1/2\pi) \int_{-\omega c}^{\omega c} d\omega\; e^{-ik\omega/\omega c}\, e^{+i\omega t} = (1/2\pi) \int_{-\omega c}^{\omega c} d\omega\; e^{-i(k/\omega c - t)\omega}$$

$$= (1/2\pi) \int_{-\omega c}^{\omega c} d\omega\; \cos[(k/\omega_c - t)\omega] = (1/\pi) \int_{0}^{\omega c} d\omega \cos[(k/\omega_c - t)\omega]$$

Maple gives this integral as

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int(cos((k*(1/wc)-t)*w), w=0..wc);
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$$\frac{wc\,(-\sin(k)\cos(t\,wc) + \cos(k)\sin(t\,wc))}{-k + t\,wc}$$

$$= \omega_c [\sin(\omega_c t)\cos(k) - \cos(t\omega_c)\sin(k)]/(\omega_c t - k)$$

$$= \omega_c [\sin(\omega_c t - k)]/(\omega_c t - k) = \omega_c \operatorname{sinc}(\omega_c t - k)$$

The original δ(t) had peak at t = 0, this thing has peak at $t = k/\omega_c$. This result agrees more or less with Pam page 10. We see a time delay here of $\Delta t = k/\omega_c$ and this is then the "group delay".

Meanwhile, the filter was $e^{-ik\omega/\omega_c} = e^{-i\varphi(\omega)}$ so it had $\varphi(\omega) = k\omega/\omega_c$ and $d\varphi/d\omega = k/\omega_c$ and in this case things work our right. So there is your "n=2" case.

Now make it a little more complicated. replace $e^{-ik\omega/\omega c}$ by $e^{-i\varphi(\omega)}$. Then we get

$$u(t) = (1/2\pi) \int_{-\omega c}^{\omega c} d\omega \; e^{-i\varphi(\omega)} e^{+i\omega t}$$

Now maybe claim that ω_c is small, so ω is always near ω_c in this integral. We then can do

$$(1/2\pi) \int_{-\omega c}^{\omega c} d\omega \; e^{-i[\varphi(0)+\varphi'(0)\omega]} e^{+i\omega t} = e^{-i\varphi(0)} \int_{-\omega c}^{\omega c} d\omega \, e^{-i\varphi'(0)\omega} e^{+i\omega t}$$

where now $\varphi'(0)$ plays the role of k/ω_c before. We will then end up with $T_d = \varphi'(0)$ and this then shows that the group delay is approximately $T_d = \varphi'(0)$. The narrower the frequency band and the smoother $\varphi(\omega)$ is, the more this is true.

Plan G. Let's now try a slight generalization. Take

$G(\omega) = |G(\omega)|e^{-i\varphi(\omega)}$ in a narrow passband $(\omega_2\text{-}a, \omega_2+a)$, G = 0 elsewhere.

$x(t) = \delta(t)$ as before

Then we get

$$u(t) = (1/2\pi) \int_{\omega_2-a}^{\omega_2+a} d\omega \; |G(\omega)| \, e^{-i\varphi(\omega)} \, X_{pulse}(\omega) \, e^{+i\omega t}$$

In the narrow band we approximate with first two terms of a Taylor series

$$\varphi(\omega) \approx \varphi(\omega_2) + \varphi'(\omega_2)\omega$$

Then get

$$u(t) \approx (1/2\pi) \, |G(\omega_2)| \, X_{pulse}(\omega_2) \int_{\omega_2-a}^{\omega_2+a} d\omega \; e^{-i\varphi(\omega 2)} \; e^{-i\varphi'(\omega 2)\omega} e^{+i\omega t}$$

$$= (1/2\pi) \, |G(\omega_2)| \, X_{pulse}(\omega_2) \, e^{-i\varphi(\omega 2)} \int_{\omega_2-a}^{\omega_2+a} d\omega \; e^{+i\omega[t-\varphi'(\omega 2)]}$$

Now set $\varphi'(\omega_2) = b$. We then have this integral

$$\int_{\omega_2-a}^{\omega_2+a} d\omega \; e^{+i\omega[t-b]} = 2 \int_0^{\omega_2+a} d\omega \cos[(t-b)\omega]$$

$$= 2 \, (t-b)^{-1} \sin[(t-b)(\omega_2-a)] = 2 \, (\omega_2-a) \, \sin[\,(\omega_2-a)\,(t-b)] \; / \; [\,(\omega_2-a)\,(t-b)]$$

$= 2\ (\omega_2\text{-}a)\ \mathrm{sinc}[\ (\omega_2\text{-}a)\ (t\text{-}b)]$

The sinc function has its peak at t = b, so the group delay is $b = \varphi'(\omega_2)$, there you are! In total we get

$$u(t) \approx (1/\pi)\ (\omega_2\text{-}a)\ |G(\omega_2)|\ X_{\mathbf{pulse}}(\omega_2)\ e^{-i\varphi(\omega 2)}\ \mathrm{sinc}[\ (\omega_2\text{-}a)\ (t\text{-}b)]$$

OK, I have now written this up as part (b) of Section 21. It is OK.