Properties of the Fourier Transform PhL 4.6.13

All incorporated into Appendix C.

Plan for Appendix C on Fourier Details.

1. Introduce the ^ notation, and show basic properties of fourier such as (not sure of these)

(a) $(f\hat{}(s))\hat{}^{-1}(u) = (f\hat{}^{-1}(s))\hat{}(u) = f(u)$ $\qquad \mathcal{F}^{-1}\{\mathcal{F}[f(t),s], u\} = \mathcal{F}\{\mathcal{F}^{-1}[f(t),s], u\}$

or

$(f\hat{})\hat{}^{-1} = (f\hat{}^{-1})\hat{} = f$ $\qquad \mathcal{F}^{-1}\mathcal{F}\, f = \mathcal{F}\mathcal{F}^{-1}\, f = f$

(b) $f\hat{}^{-1}(s) = (1/2\pi)\, f\hat{}(-s)$ $\qquad \mathcal{F}^{-1}[f(u),s] = (1/2\pi)\, \mathcal{F}[f(-u),s]$

2. Do the 1/ω transform and recovery three different ways
   a) regular fourier of 1/ω with principle part contour, get the sgn thing
   b) regular fourier of 1/(ω±iε) with recovery, get θ(t)
   c) generalized fourier with 1/ω, and recovery.

3. Do the Hilbert Transform as a convolution, making use of result 2(a) above

$$\mathcal{H}(X) \equiv +(1/\pi) \fint_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega-\omega'} = \mathcal{H}[X(\omega'),\omega]$$

which write as

$$f^{h}(\omega) = (1/\pi) \fint_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{\omega-\omega'} = \mathcal{H}[X(\omega'),\omega]$$

or

$$f^{h}(t) = (1/\pi) \fint_{-\infty}^{\infty} d\omega \frac{f(\omega)}{t-\omega} = \mathcal{H}[f(x),t]$$

Now start with this equation

$$f^{h}(t) = (1/\pi) \fint_{-\infty}^{\infty} d\omega \frac{f(\omega)}{t-\omega} = (1/\pi) \fint_{-\infty}^{\infty} dt' \frac{1}{t-t'} f(t')$$

Treat this as a convolution equation of the form

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad a\hat{}(\omega) = b\hat{}(\omega)\, c\hat{}(\omega)$$

Thus we have

$$f^{h}(t) = \fint_{-\infty}^{\infty} dt' \frac{(1/\pi)}{t-t'} f(t') \qquad \Leftrightarrow \qquad (f^{h})\hat{}(\omega) = \left[\frac{(1/\pi)}{t}\right]\hat{}(\omega)\, f\hat{}(\omega)$$

But from below we know that

$$\left[\frac{(1/\pi)}{t}\right]\hat{}(\omega) = -i\,\mathrm{sgn}(\omega)$$

Therefore we end up with

$$(f^{\mathbf{h}})\hat{}(\omega) = -i\,\mathrm{sgn}(\omega)\, f\hat{}(\omega)$$

and this agrees with my PDF. Maybe in terms of $X(\omega)$ and $x(t)$ write this as

$$(f^{\mathbf{h}})\hat{}(\omega) = -i\,\mathrm{sgn}(\omega)\, X(\omega)$$

which is almost the same as in the PDF and probably wiki.

Question: Is this true or not?

$$(f^{\mathbf{h}})\hat{}(\omega) = (\hat{f})^{\mathbf{h}}(\omega)$$

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First, I have some notation problems, so let's start with my Spectral conventions,.

Fourier Integral Transform:

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

What other notations might be useful?

$$X(\omega) = \mathcal{L}[x(t), \omega] \quad \text{or} \quad X = \mathcal{L}[x]$$

$$x(t) = \mathcal{L}^{-1}[X(\omega), t] \quad \text{or} \quad x = \mathcal{L}^{-1}[X]$$

Then we find that

$$x = \mathcal{L}^{-1}[X] = \mathcal{L}^{-1}[\mathcal{L}[x] = \mathcal{L}^{-1}\mathcal{L}\,[x] = 1\,[x] = x$$

In the longer notation how does this look?

$$x(t) = \mathcal{L}^{-1}[X(\omega), t] = \mathcal{L}^{-1}[\mathcal{L}[x(t), \omega], t] = \text{it must be } x(t)$$

In this last situation, I see no way to get $\mathcal{L}^{-1}\mathcal{L} = 1$. Maybe this

$$\langle t|x\rangle = \langle t|\, \mathcal{L}^{-1}[X(\omega)] \rangle = \langle t|\, \mathcal{L}^{-1}[\langle \omega|X\rangle] \rangle = \langle t\,| \ ..???$$

Go back and write

$$x = \mathcal{L}^{-1}X = \mathcal{L}^{-1}\mathcal{L}x = 1x = x$$

So here x and X are function vectors in a function Hilbert Space.

Theorem 1.

$$\mathcal{L}^{-1}[X(\omega), t\,] = x(t)$$

$$\mathcal{L}^{-1}[X(u), t\,] = (1/2\pi)\,\mathcal{L}[X(u),-t]$$

$$\mathcal{L}[X(t),\omega] = \int_{-\infty}^{\infty} dt\, X(t)\, e^{-i\omega t}$$

$$\mathcal{L}[X(u),\omega] = \int_{-\infty}^{\infty} du\, X(u)\, e^{-i\omega u}$$

$$\mathcal{L}[X(u),t] = \int_{-\infty}^{\infty} du\, X(u)\, e^{-itu} = f(t)$$

$$\mathcal{L}^{-1}[X(\omega), t\,] = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t}$$

$$\mathcal{L}^{-1}[X(u), t\,] = (1/2\pi) \int_{-\infty}^{\infty} du\, X(u)\, e^{+iut} = g(t)$$

$$\mathcal{L}^{-1}[X(u), t\,] = (1/2\pi) \int_{-\infty}^{\infty} du\, X(-u)\, e^{-iut} = h(t)$$

$$\mathcal{L}[X(-u),t] = \int_{-\infty}^{\infty} du\, X(-u)\, e^{-itu} = k(t)$$

Clearly we have

$g(t) = f(-t)$.

Therefore

$$\mathcal{L}^{-1}[X(u), t] = (1/2\pi)\, \mathcal{L}[X(u),-t]$$

Secondly we have

$$h(t) = (1/2\pi)k(t)$$

Therefore

$$\mathcal{L}^{-1}[X(u), t] = (1/2\pi)\, \mathcal{L}[X(-u),t]$$

So we can combine into one statement

$$\mathcal{L}^{-1}[f(u), t] = (1/2\pi)\, \mathcal{L}[f(u),-t] = (1/2\pi)\, \mathcal{L}[f(-u),t]$$

Now I can apply the above theorem

$$\mathcal{L}^{-1}[X(\omega), t] = (1/2\pi)\, \mathcal{L}[X(\omega),-t] = (1/2\pi)\, \mathcal{L}[X(-\omega),t]$$

$$\mathcal{L}^{-1}[x(t), \omega] = (1/2\pi)\, \mathcal{L}[x(t),-\omega] = (1/2\pi)\, \mathcal{L}[x(-t),\omega]$$

Theorem 2. suppose $x(t) \leftrightarrow X(\omega)$ where $X = \mathcal{F}x$. Then conjecture that

$$X(t) \leftrightarrow x(-\omega) \qquad \text{and} \qquad X(-t) \leftrightarrow x(\omega)$$

Proof:

Start with from above

$$X(\omega) = \mathcal{L}[x(u), \omega]$$

Maybe I really need a better overall notation like Stak:

$$f\hat{}(\omega) = \mathcal{L}[f(u), \omega]$$

Plan B: start all over

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

Rewrite the above using a generic function f

$$f\hat{}(\omega) = \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} \qquad \text{projection = transform} \qquad (1.1)$$

$$f\hat{}^{-1}(t) = (1/2\pi) \int_{-\infty}^{\infty} du\, f(u)\, e^{+iut} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

It seems pretty clear that

$$f\hat{}^{-1}(-\omega) = (1/2\pi) \int_{-\infty}^{\infty} du\, f(u)\, e^{-iu\omega} = (1/2\pi)\, f\hat{}(\omega) \qquad \text{Theorem 1}$$

So we are dealing here with three distinct functions: f(u), f^(u) and f^$^{-1}$(u). But we just showed that

$$f\hat{}^{-1}(u) = (1/2\pi)\, f\hat{}(-u)$$

so two of these three functions have a simple relationship.

Example 1:

Projection: Let f(u) = 1/u. Then we have tick addable since we go right through the pole as limit.

$$(1/u)\hat{}(\omega) = \int_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} = \fint_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u}$$

We can evaluate the integral this way

$$\fint_{-\infty}^{\infty} du(1/u)\, e^{-i\omega u} = \fint_{-\infty}^{\infty} du(1/u)\, [-i\sin(\omega u)] = (-i2) \fint_{0}^{\infty} du\, \sin(\omega u)/u$$

$$= (-2i)\, (\pi/2)\, \text{signum}(\omega) = -i\pi\, \text{sgn}(\omega)$$

Therefore

$$(1/u)\hat{}(\omega) = -i\pi\, \text{sgn}(\omega)$$

or

$$(1/t)\hat{}(\omega) = -i\pi\, \text{sgn}(\omega)$$

But is this really a Fourier Transform? I am not sure.

Inversion: How does the recovery work?

$$(1/2\pi) \int_{-\infty}^{\infty} d\omega\, (1/t)\hat{}(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, [-i\pi\text{sgn}(\omega)]\, e^{+i\omega t}$$

$= (1/2\pi)(-i\pi) \int_{-\infty}^{\infty} d\omega\ \mathrm{sgn}(\omega)\ e^{+i\omega t} = (-i/2) \int_{-\infty}^{\infty} d\omega\ \mathrm{sgn}(\omega)\ [i \sin(\omega t]$

$= (-i/2)\ i\ 2 \int_{0}^{\infty} d\omega\ \sin(\omega t) = \int_{0}^{\infty} d\omega\ \sin(\omega t) = -(1/t)\cos(\omega t)|_{0}^{\infty} = -(1/t)\ [\cos(\infty t) - \cos(0)]$

$= (1/t)$

The distributional trick is to set $\cos(\infty t) = 0$. In more detail,

$$\int_{0}^{\infty} d\omega\ \sin(\omega t) = \lim_{\varepsilon \to 0} \left[\int_{0}^{\infty} d\omega\ \sin(\omega t)\ e^{-\varepsilon \omega} \right] = \lim_{\varepsilon \to 0} \frac{t}{t^2+\varepsilon^2} = \frac{1}{t}$$

Example 2:

Projection: Let $f(u) = \frac{1}{u \pm i\varepsilon}$. Now our Fourier projection misses the pole. We find

$$\left(\frac{1}{u \pm i\varepsilon}\right)\hat{}(\omega) = \int_{-\infty}^{\infty} du \frac{1}{u \pm i\varepsilon}\ e^{-i\omega u} = \oint_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u} \mp i\pi$$

The principle value integral is this

$$\oint_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u} = \oint_{-\infty}^{\infty} du(1/u)[\ -i\sin(\omega u)] = (-i)\ 2 \int_{\mathbf{0}}^{\infty} du\ (1/u)\sin(\omega u)$$

$= (-2i)\ \pi/2\ \mathrm{sgn}(\omega) = -i\pi\ \mathrm{sgn}(\omega)$ // which is just $(1/t)\hat{}(\omega)$ computed above

Therefore

$$\left(\frac{1}{u \pm i\varepsilon}\right)\hat{}(\omega) = -i\pi\ \mathrm{sgn}(\omega) \mp i\pi = -i\pi\ (\mathrm{sgn}(\omega) \pm 1)$$

Assume first the upper signs

$$\left(\frac{1}{u+i\varepsilon}\right)\hat{}(\omega) = -i\pi\ \mathrm{sgn}(\omega) - i\pi = -i\pi\ (\mathrm{sgn}(\omega) + 1)$$

If $\omega > 0$, the result is $-2\pi i$, and if $\omega < 0$ the result is 0. So

$$\left(\frac{1}{u+i\varepsilon}\right)\hat{}(\omega) = -2\pi i\theta(\omega)$$

Now assume the lower signs

$$\left(\frac{1}{u-i\varepsilon}\right)^{\wedge}(\omega) = -i\pi\,\mathrm{sgn}(\omega) + i\pi = -i\pi\,(\mathrm{sgn}(\omega) - 1)$$

If $\omega > 0$, the result is 0. If $\omega < 0$, the result is $+2\pi i$. So

$$\left(\frac{1}{u-i\varepsilon}\right)^{\wedge}(\omega) = 2\pi i\theta(-\omega)$$

Combining these results we get

$$\left(\frac{1}{u\pm i\varepsilon}\right)^{\wedge}(\omega) = \mp\, 2\pi i\, \theta(\pm\omega)$$

Inversion:

$$(1/2\pi) \int_{-\infty}^{\infty} d\omega \left(\frac{1}{u\pm i\varepsilon}\right)^{\wedge}(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, [\mp 2\pi i\, \theta(\pm\omega)]\, e^{+i\omega t}$$

$$= (1/2\pi)(\mp 2\pi i) \int_{-\infty}^{\infty} d\omega\, \theta(\pm\omega)\, e^{+i\omega t} = (\mp\, i) \int_{-\infty}^{\infty} d\omega\, \theta(\pm\omega)\, e^{+i\omega t}$$

First take the upper sign

$$= (-\, i) \int_{0}^{\infty} d\omega\, e^{+i\omega t}$$

If t has a small positive imaginary part, the integral converges to (-1/it) to give

$$= (-\, i)\,(-1/it) = 1/t$$

but we write t as $t+i\varepsilon$ to show that it has this small positive imaginary part, so the result is

$$(1/2\pi) \int_{-\infty}^{\infty} d\omega \left(\frac{1}{u+i\varepsilon}\right)^{\wedge}(\omega)\, e^{+i\omega t} = \frac{1}{t+i\varepsilon}$$

as desired. Now we look at the lower sign

$$(+i) \int_{-\infty}^{0} d\omega\, e^{+i\omega t} = (+i) \int_{0}^{\infty} d\omega\, e^{-i\omega t}$$

If t has a small negative imaginary part, the integral converges to (+1/it) to give

$$= (+i)\,(+1/it) = 1/t = \frac{1}{t-i\varepsilon}$$

which is again the desired result.

But this is not what I expected.

Example 3: Let f(u) = θ(u). Then

$$\theta\hat{}(\omega) = \int_{-\infty}^{\infty} du\, \theta(u)\, e^{-i\omega u} = \int_{0}^{\infty} du\, e^{-i\omega u}$$

If we assumed ω had the right complex value, we could make the integral converge. We note that

$$\int_{0}^{\infty} du\, e^{-au} = 1/a$$

Then let a = iω to get

$\theta\hat{}(\omega) = 1/(i\omega)$ $\quad$ $\mathrm{Im}(\omega) < 0$
$\theta\hat{}(\omega) =$ does not exist $\quad$ $\mathrm{Im}(\omega) \geq 0$

But if ω is real, we have to change our function definition.

What we really need is

$$\mathrm{Re}(a) > 0 \quad \text{or} \quad \mathrm{Re}(i\omega) > 0 \quad \text{or} \quad \mathrm{Re}[\, i\mathrm{Re}(\omega) + i^2\mathrm{Im}(\omega)] > 0 \quad \text{or} \quad -\mathrm{Im}(\omega) > 0$$

or $\mathrm{Im}(\omega) < 0$.

Thus, we are OK if ω is in the lower half plane. We could then approach the real axis from below.

What does the inverse look like? Since we don't know $\theta\hat{}(\omega)$ for real ω, we cannot do the inversion. The FT does not exist, so we cannot invert!

Example 3: Let $f(u) = \theta(u)e^{-\varepsilon u}$ in limit $\varepsilon \to 0$. Then we find

$$(\theta_\varepsilon)\hat{}(\omega) = \frac{1}{i\omega+\varepsilon} = \frac{1}{i}\frac{1}{\omega - i\varepsilon} \qquad \text{valid for } \omega = \text{real}$$

The inversion formula then works perfectly. We find

$$[(\theta_\varepsilon(u))\hat{}]\hat{}^{-1} = \theta_\varepsilon(u)$$

Question: How do you explain or prove this idea?

$$\frac{1}{\omega \mp i\varepsilon} = pf(1/\omega) \pm i\pi\delta(\omega)$$

You have to put it inside an integral. Any integral ?

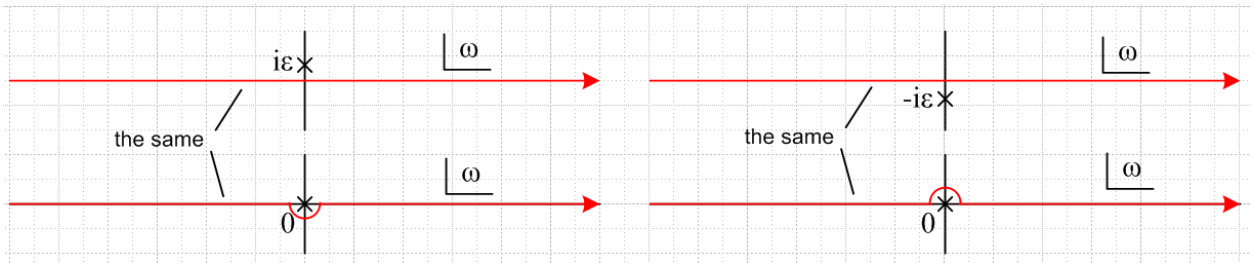
$$\int_{-\infty}^{\infty} d\omega\, f(\omega) \frac{1}{\omega - i\varepsilon} = \int_{-\infty}^{\infty} d\omega\, f(\omega)\, [pf(1/\omega) \pm i\pi\delta(\omega)]$$

$$= \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) + i\pi\, f(0)$$

Well its really easier than all this, we just need some pictures: iε

The Pole Avoidance Rule

Consider these drawings and the pair of equations below each drawing:



$$\int_{-\infty}^{\infty} d\omega\, f(\omega) \frac{1}{\omega - i\varepsilon} = \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) + i\pi\, f(0) \qquad \int_{-\infty}^{\infty} d\omega\, f(\omega) \frac{1}{\omega + i\varepsilon} = \fint_{-\infty}^{\infty} d\omega\, f(\omega)(1/\omega) - i\pi\, f(0)$$

$$\frac{1}{\omega - i\varepsilon} = pf(1/\omega) + i\pi\delta(\omega) \qquad \frac{1}{\omega + i\varepsilon} = pf(1/\omega) - i\pi\delta(\omega)$$

This is a concept that requires a lot of words. We assume that f(ω) is such that the integrals converge and f(ω) is well defined at ω = 0. We are interested only in the limit ε→0.

Left Side. Consider the top left red arrow contour. If we try to take ε→0, the pole moves down and hits the contour, which is a poorly defined thing to happen. To prevent this from happening, we first make a tiny semi-circular deformation of the contour so it goes around ω = 0. One is certainly allowed to deform a contour and not change an integral, as long as the deformation hits no singularities. After doing this "for free" deformation, we *then* let the ε→0 so the pole moves down to the real axis. If we now evaluate the integral, we get two famous pieces: The first piece is the **principle value integral** which is this,

$$\fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) \equiv \lim_{\alpha\to 0} \left[\int_{-\infty}^{-\alpha} + \int_{\alpha}^{\infty} \right] d\omega\, f(\omega)\, (1/\omega)$$

The second piece is a half-circle contour around the pole which gives *one half* the pole residue which result is then $(1/2)\, 2\pi i\, f(0) = i\pi f(0)$. In general, if a contour goes some percentage around a pole, it picks up that percentage of the reside, for this reason, letting $\omega = Re^{i\theta}$,

$$\oint \frac{d\omega}{\omega} = \oint \frac{Re^{i\theta} i d\theta}{Re^{i\theta}} = i \int_{\theta_1}^{\theta_2} d\theta = i(\theta_2 - \theta_1).$$

If we go half way around the pole, then $i(\theta_2-\theta_1) = i\pi$. So we have now derived the top left equation. Recall from Appendix A that a distributional equation is one which gains its meaning when placed inside an integral. So consider

$$\int_{-\infty}^{\infty} d\omega\, f(\omega) \frac{1}{\omega - i\varepsilon} = \int_{-\infty}^{\infty} d\omega\, f(\omega) [\ pf(1/\omega) + i\pi\delta(\omega)\ ] = \int_{-\infty}^{\infty} d\omega\, f(\omega)\, pf(1/\omega) + i\pi\, f(0)$$

Here both $pf(1/\omega)$ and $\delta(\omega)$ are symbolic functions as discussed in Appendix A. The meaning of $\delta(\omega)$ seems clear (sifting property) while the meaning of $pf(1/\omega)$ is precisely this:

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, pf(1/\omega) = \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega)$$

Officially $f(\omega)$ should be a distribution theory "test function", but we just take it to be any reasonable function as described above. So we have now derived both equations on the left.

Right Side. This is the same idea, but the required contour deformation is different, and since the semicircle then goes clockwise around the pole, we pick up minus half the residue which is $-i\pi f(0)$. Now both equations on the right are derived. We have then proven

The Pole Avoidance Rule:

$$\frac{1}{\omega \mp i\varepsilon} = pf(1/\omega) \pm i\pi\delta(\omega)$$

Notice that this rule has no connection with phase sign conventions of the Fourier Transform. It really has nothing at all to do with the Fourier Transform in fact.

The lower left equation is just such a distributional equation, and its meaning is nothing more and nothing less than what is stated right above it. The $i\pi\delta(\omega)$ part seems pretty clear, and here is the meaning of the $pf(1/\omega)$ part

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, pf(1/\omega) = \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) = \text{the limit shown above}$$

Stakgold uses this notation pf($1/\omega$) where pf just means "pseudofunction" which is a generic name for a distributional symbolic function like $\delta(x)$.

Left Side. We just repeat the above discussion, but this time the contour is deformed upward first, and then the pole is allowed to move up. Since the contour now goes half way around the pole in the clockwise direction, the little semicircle picks up minus half the residue, or $-i\pi f(0)$.