(d) application to coaxial cable

Consider an infinitely long coaxial cable driven at its left end at z = 0. A coaxial cable acts as a filter G(ω). In the frequency domain, it we drive the cable with I(ω), the output is O(ω) where

$$O(\omega, z) = G(\omega, z)\, I(\omega) \qquad G(\omega) = e^{-\gamma(\omega) z} \qquad \gamma(\omega) = \alpha(\omega) + i\beta(\omega)$$

The transfer function G(ω,z) is like our X(ω) above, and satisfies the dispersion relations shown. In particular, it satisfies (31.9),

$$\alpha(\omega) = \alpha(\infty) + (2/\pi) \int_0^\infty d\omega' \omega' \frac{\beta(\omega')}{\omega'^2 - \omega^2} \qquad \textit{(31.9a)}$$

$$\beta(\omega) = \beta(\infty) - (2/\pi)\ \omega \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2 - \omega^2} \qquad \textit{(31.9b)}$$

We have just argued that if α(ω) is slowly varying near ω, then $K(\omega) = (2/\pi) \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2 - \omega^2}$ is roughly constant κ, and so $\tau(\omega) = d[\,\omega K(\omega)\,]/d\omega \approx \kappa$ and there is very little dispersion. One way to get α(ω) to be slowly varying is from (31.9a) in the case β(ω') ≈ 0. In this case (31.9a) says α(ω) ≈ α(∞) = constant. Thus, in a region where γ(ω) has a very small (or no) imaginary part, we expect to get small (or no) dispersion of pulses.

So how does α(ω) vary with ω in a coaxial cable? We first show that γ(ω) is basically the same as the index of refraction n(ω) for the dielectric material in the cable. For a coaxial cable one has

$$\gamma = \sqrt{(R+i\omega L)(G+i\omega C)}$$

where R,L,G and C are resistance, inductance, conductance (across the dielectric) and capacitance all per unit length of the cable. If we ignore R and G, we find that

$$\gamma(\omega) \approx i\,\omega\sqrt{LC} \qquad \Rightarrow \qquad \alpha(\omega) = 0 \qquad \beta(\omega) = \omega\sqrt{LC}$$

The capacitance C of a section of coaxial cable is given by $C = \varepsilon(\omega) 2\pi\varepsilon_0 / \ln(b/a) = k_1\, \varepsilon(\omega)$, where ε(ω) is the dielectric "constant", which in general is not constant as a function of ω. Thus we have

$$\beta(\omega) = \omega\sqrt{LC} = \omega\sqrt{Lk_1}\,\sqrt{\varepsilon(\omega)}$$

If ε(ω) really were a constant, then we would have linear phase and constant group delay.

From Maxwell's equations one knows that the index of refraction is given by

$$n(\omega) = \sqrt{\mu\varepsilon} = \sqrt{\varepsilon(\omega)}\ /k_2$$

Then

$$\beta(\omega) = \omega \sqrt{L k_1}\, k_2\, n(\omega)$$

so $\gamma(\omega)$ is directly related to the index of refraction $n(\omega)$.

One way to get $\alpha(\omega)$ to be slowly varying is from the first equation above in the case $\mathrm{Im}(\varepsilon(\omega)) \approx 0$ meaning $\beta(\omega') \approx 0$. In this case the first equation says $\alpha(\omega) \approx \alpha(\infty)$ = constant. Thus, in a region where the index $\varepsilon(\omega)$ has a very small (or no) imaginary part, we expect to get small (or no) dispersion of pulses.

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For an infinitely long coaxial cable of length driven at one end by some voltage $V_o e^{i\omega t}$, the voltage distance z down the cable is given by

$$V(z,t;\omega) = G(\omega)\, V_0\, e^{i\omega t}$$

$$V(z,t) = V_0 e^{-\gamma(\omega) z}\, e^{i\omega t} = V_0\, e^{-\alpha(\omega) z}\, e^{-i\beta(\omega) z}\, e^{i\omega t}$$

where $\alpha(\omega)$ is the attenuation factor and $\beta(\omega)$ is the phase. The complex factor $\gamma(\omega)$ is given by

$$\gamma = \sqrt{(R+i\omega L)(G+i\omega C)}$$

where R,L,G and C are resistance, inductance, conductance (across the dielectric) and capacitance all per unit length of the cable.

The functions $\alpha(\omega)$ and $\beta(\omega)$ satisfy dispersion relations of the form (31.9)

$$\alpha(\omega) = \alpha(\infty) + (2/\pi) \int_0^\infty d\omega' \omega' \frac{\beta(\omega')}{\omega'^2 - \omega^2} \qquad \textit{(31.9a)}$$

$$\beta(\omega) = \beta(\infty) - (2/\pi)\, \omega \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2 - \omega^2} \qquad \textit{(31.9b)}$$

We have just argued that if $\alpha(\omega)$ is slowly varying near ω, then $K(\omega) = (2/\pi) \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2 - \omega^2}$ is roughly constant κ, and so $\tau(\omega) = d[\,\omega K(\omega)\,]/d\omega \approx \kappa$ and there is very little dispersion. One way to get $\alpha(\omega)$ to be slowly varying is from the first equation above in the case $\mathrm{Im}(\varepsilon(\omega)) \approx 0$ meaning $\beta(\omega') \approx 0$. In this case the first equation says $\alpha(\omega) \approx \alpha(\infty)$ = constant. Thus, in a region where the index $\varepsilon(\omega)$ has a very small (or no) imaginary part, we expect to get small (or no) dispersion of pulses.

$$V = V_o e^{i\omega t}\ e$$

then it turns out the dispersion relations

This effect is also known as "dispersion". It happens that $n(\omega) = \sqrt{\varepsilon(\omega)}$, where $\varepsilon(\omega)$ is called the dielectric constant of the medium, although it is sometimes not very "constant" as a function of ω.

It turns out that $\varepsilon(\omega)$ has the right properties to satisfy the above pair of equations (31.9) where now $\varepsilon(\omega) = e^{-\gamma(\omega)} = e^{-\alpha(\omega)}\ e^{-i\beta(\omega)}$:

$$\alpha(\omega) = \alpha(\infty) + (2/\pi) \int_0^\infty d\omega' \omega' \frac{\beta(\omega')}{\omega'^2-\omega^2} \qquad (31.9a)$$

$$\beta(\omega) = \beta(\infty) - (2/\pi)\ \omega \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2-\omega^2} \qquad (31.9b)$$

We have just argued that if $\alpha(\omega)$ is slowly varying in some region, then $K(\omega) = (2/\pi) \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2-\omega^2}$ is roughly constant κ, and so $\tau(\omega) = d[\ \omega K(\omega)\ ]/d\omega \approx \kappa$ and there is very little dispersion. One way to get $\alpha(\omega)$ to be slowly varying is from the first equation above in the case $\mathrm{Im}(\varepsilon(\omega)) \approx 0$ meaning $\beta(\omega') \approx 0$. In this case the first equation says $\alpha(\omega) \approx \alpha(\infty)$ = constant. Thus, in a region where the dielectric constant $\varepsilon(\omega)$ has a very small (or no) imaginary part, we expect to get small (or no) dispersion of pulses.

In effect, a dielectric medium is a filter, and $\varepsilon(\omega)$ is the transfer function of the filter. The input and output of this "filter" are known as the "electric field" E and the "electric displacement" D. In the frequency domain the equation of this filter is (21.2) which here reads (at any point in space)

$$D(\omega) = \varepsilon(\omega)\ E(\omega)\ .$$

If $\varepsilon(\omega)$ has a small imaginary part in some range of ω, this filter is non-dispersive.

For a transmission line like a coaxial cable, if we ignore conductance G through the dielectric and we ignore inductance L along the cable, if the cable is driven with voltage V_0 at one end $z = 0$, the voltage at the other end is given by

$$V(z,t) = V_0 e^{-\gamma(\omega) z}\ e^{i\omega t} = V_0\ e^{-\alpha(\omega) z}\ e^{-i\beta(\omega) z}\qquad e^{i\omega t}$$

The attenuation $\alpha(\omega)$ and phase $\beta(\omega)$ satisfy the dispersion relations (31.9) above and we reach the same conclusion.

Away from electromagnetic resonances of the medium, $\varepsilon(\omega)$ has a very small imaginary part for a non-polar dielectric like polyethylene or teflon. Thus, if we could ignore ohmic losses in the conductors, coaxial cables using these materials as dielectrics would be non-dispersive up to infrared frequencies -- where vibrational and rotational resonances set in.