Note that page numbering is turned on in this template.

29. Where do Image Spectra come from?

Consider the amplitude-modulated pulse train and its Fourier Integral spectrum from summary box (25.4). The pulse shape is $x_{pulse}(t)$ and the pulse spacing is T_1.

$$w(t) = \sum_{n=-\infty}^{\infty} y_n \, x_{pulse}(t - t_n) \qquad (25.1) \qquad (29.1)$$

$$W(\omega) = (1/T_1) X_{pulse}(\omega) \, Y'(\omega) \quad . \qquad (25.3) \qquad (29.2)$$

In the last equation, $Y'(\omega)$ is the Digital Fourier Transform of the sequence of y_n values which provide the modulation of the pulses,

$$Y'(\omega) \equiv T_1 \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega t_n} \qquad \text{projection} = \text{transform} \qquad (22.2) \qquad (29.3)$$

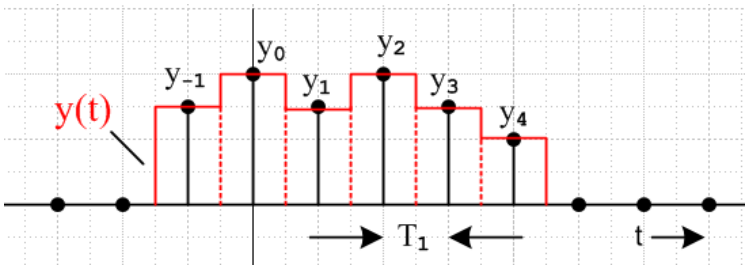
We assume now that the samples y_n are of some continuous signal y(t) which has Fourier Integral spectrum $Y(\omega)$. Then the relationship between $Y'(\omega)$ and $Y(\omega)$ is given by

$$Y'(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) = [\, Y(\omega) + \sum_{m \neq 0} Y(\omega - m\omega_1) \,] \quad . \qquad (23.1) \qquad (29.4)$$

We noted that $Y'(\omega)$ therefore has some sort of central region of shape $Y(\omega)$ and an infinite set of **image spectra** going off to the left and right of this central region, spaced by ω_1.

A priori, we don't know exactly what $Y(\omega)$ looks like because we don't know y(t), we only know its sample values y_n. In order to have something to work with, we have to assume some shape for y(t). This shape, whatever it might be, is completely independent of shape of $x_{pulse}(t)$ in (29.1).

For purposes of discussion, then, let us assume that the signal y(t), whose samples are y_n, is a stair-step function which just tracks the y_n values as in this example,



But *this* is an amplitude-modulated finite pulse train with $x'_{pulse}(t)$ = a box of height A = 1 and width $\tau = T_1$. The Fourier Integral Transform spectrum Y(ω) associated with just the box centered at t = 0 we know from (9.2) is

$$y_0(T_1)\ \mathrm{sinc}(\omega T_1/2).$$

The spectrum of the next box is, according to the time translation rule (12.1),

$$y_1(T_1)\ \mathrm{sinc}(\omega T_1/2)\ e^{-i\omega T_1}$$

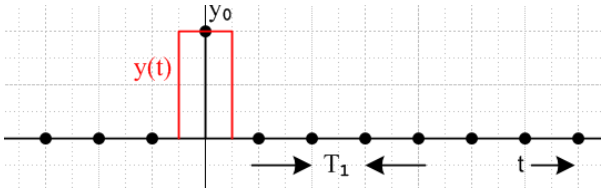
and by superposition we find that

$$\begin{aligned} Y(\omega) &= (T_1)\ \mathrm{sinc}(\omega T_1/2)\ \Sigma_n\ y_n\ e^{-in\omega T_1} \\ &= (T_1)\ \mathrm{sinc}[\pi(\omega/\omega_1)]\ \Sigma_n\ y_n\ e^{-i2\pi(n\omega/\omega_1)}\ . \\ &= \mathrm{sinc}[\pi(\omega/\omega_1)]\ Y'(\omega) \end{aligned} \qquad (29.5)$$

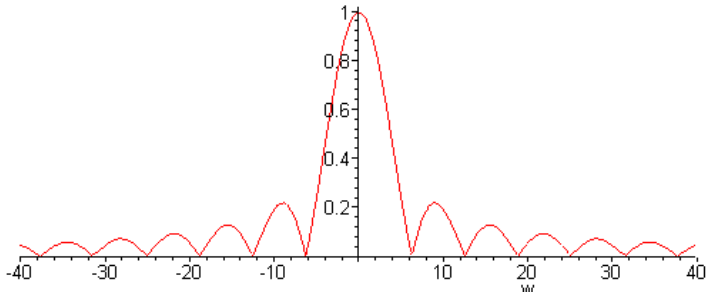
in agreement with (25.3) for the spectrum of an amplitude-modulated pulse train with x'_{pulse} = box. As a function of ω, the sum part Y'(ω) of Y(ω) is a periodic function of period ω_1 repeating out to $\omega = \pm\infty$, but the sinc function tames the spectrum so that Y(ω) is localized to a region around ω = 0 of half-width ω_1.

We shall now explore the above set of equations for some simple pulse train examples.

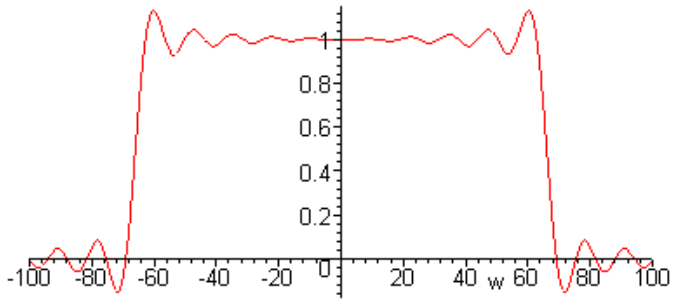
Example 1: Even the simplest case is interesting. The y_n sequence is taken as a unit impulse scaled by y_0



Then $|Y(\omega)| = y_0(T_1)\ |\mathrm{sinc}(\omega T_1/2)|$ which we plot for T_1= 1 so $\omega_1 = 2\pi$, and ω in (-40,40),



Meanwhile, we know from (29.3) that $Y'(\omega) \equiv y_0T_1$. In this example, how exactly does the image spectrum equation $Y'(\omega) = [\ Y(\omega)\ + \Sigma_{m\neq 0}Y(\omega - m\omega_1)\ ]$ work out? First, suppose we plot this Y'(ω) sum limiting the sum range to m = -100 to 100 , using $Y(\omega) = y_0(T_1)\ \mathrm{sinc}(\omega T_1/2)$ for Y(ω) :



It appears that the shifted sinc functions are adding up to produce the constant $Y'(\omega) \equiv T_1 = 1$. In terms of the math, this must mean that

$$Y'(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) = y_o T_1 \sum_{m=-\infty}^{\infty} \text{sinc}[\pi(\omega/\omega_1 - m)] = y_o T_1 \qquad (29.6)$$

which implies the following unusual sum rule, valid for any real x :

$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x-m)] = 1 \, . \qquad (29.7)$$

This result is sometimes quoted with x = 0 as $\Sigma_{m=-\infty}^{\infty} \text{sinc}(\pi m) = 1$, but it is in fact valid for any x. We have in effect proven the sum rule with the above analysis, but as usual we would like to find verification. We first process the sum as follows

$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x-m)] = \sum_{m=-\infty}^{\infty} \sin[\pi(x-m)]/[\pi(x-m)] = (1/\pi)\sin[\pi x] \sum_{m=-\infty}^{\infty} (-1)^m / (x-m).$$

The sum on the right can be further processed,

$$\sum_{m=-\infty}^{\infty} (-1)^m / (x-m) = 1/x + \left[\sum_{m=-\infty}^{-1} + \sum_{m=1}^{\infty} \right] (-1)^m / (x-m) = 1/x + 2x \sum_{m=1}^{\infty} (-1)^m \frac{1}{x^2-m^2} \, .$$

According to GR 142.3 (page 44),

3. $$\operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2}$$ (see also **1.217** 2) AD (6495.4)a

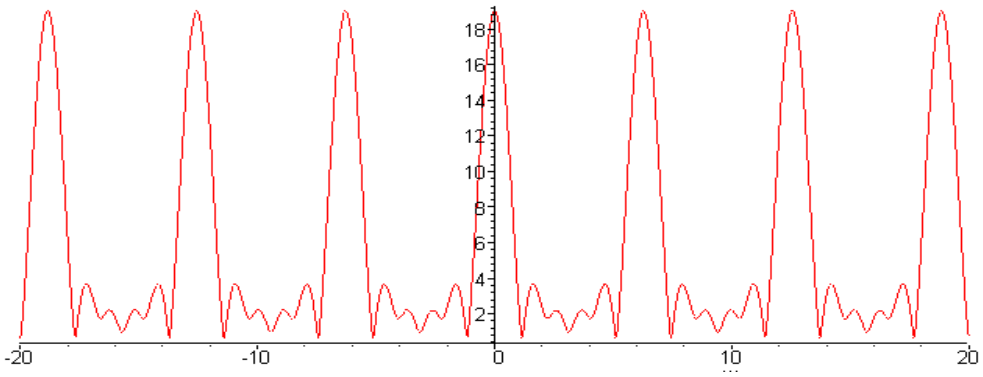
we may replace

$$1/x + 2x \sum_{m=1}^{\infty} (-1)^m \frac{1}{x^2-m^2} = \pi \csc(\pi x)$$

and then we find that

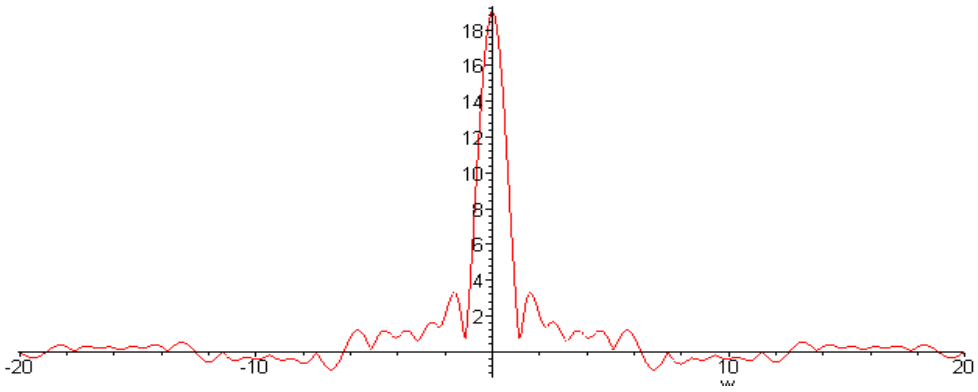
$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x-m)] = (1/\pi)\sin[\pi x]\ \pi \csc(\pi x) = 1 .$$

Example 2: Here we use the exact sample sequence shown in Figure *** above. We start here with a plot of $|Y'(\omega)| = |\Sigma_n\ y_n\ e^{-in\omega T_1}|$ of (29.3) [$T_1 = 1$ so $\omega_1 = 2\pi$, and ω in (-20,20)],



As expected, the pattern of peaks goes on forever in each direction. At $\omega = 0$ the real peak is at 19 since this is the sum of the y_n values in the figure.

The corresponding $Y(\omega)$ of (29.5) then looks like this,



which is localized roughly to the range $(-\omega_1,\omega_1) = (-2\pi,2\pi)$. There are no image spectra in $Y(\omega)$. On the other hand, $Y'(\omega)$ does have image spectra as the previous figure shows.

Now in this example, how does the equation $Y'(\omega) = [\ Y(\omega)\ + \Sigma_{m\neq 0} Y(\omega - m\omega_1)\ ]$ work out? Using (29.5) we write

$$Y'(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} \text{sinc}[\pi([\omega - m\omega_1]/\omega_1)]\ Y'(\omega - m\omega_1)$$

$$= \sum_{m=-\infty}^{\infty} \text{sinc}[\pi([\omega - m\omega_1]/\omega_1)]\ Y'(\omega) \qquad // Y'(\omega - m\omega_1) = Y'(\omega) \text{ from (22.3)}$$

$$= Y'(\omega) \sum_{m=-\infty}^{\infty} \text{sinc}[\pi([\omega - m\omega_1]/\omega_1)] = Y'(\omega) * 1 = Y'(\omega)$$

where we used (29.7) that the sinc sum is 1. Therefore, if we implement $Y(\omega) + \Sigma_{m\neq 0} Y(\omega - m\omega_1)$ using the second figure for $Y(\omega)$, we will obtain exactly the first figure for $Y'(\omega)$.

where, once again, $x_{\mathbf{pulse}}(t)$ is some arbitrary pulse shape. For any physical pulse shape $x_{\mathbf{pulse}}(t)$, the spectrum $X_{\mathbf{pulse}}(\omega)$ will be localized in some region around $\omega = 0$, and the fact that $Y'(\omega)$ has an infinite set of image spectra does not cause $W(\omega)$ to have image an infinite set of image spectra. It might be that $W(\omega)$ has some *finite* number of image spectra if $X_{\mathbf{pulse}}(\omega)$ is very broad.

However, in the idealized limit that $x_{\mathbf{pulse}}(t) = \delta(t)$, in which case $X_{\mathbf{pulse}}(\omega) = 1$, we find that

$$W(\omega) = (1/T_1)\ Y'(\omega)$$

and in this limiting case (only), $W(\omega)$ inherits the full infinite set of image spectra of $Y'(\omega)$. This means that, when considered in the frequency domain, our pulse train

$$w(t) = \sum_{n=-\infty}^{\infty} y_n\ x_{\mathbf{pulse}}(t - t_n)$$

must have infinite energy.

Question: Suppose we have an infinite pulse train with all $y_n = 1$ and $x_{\mathbf{pulse}}$= box width T_1. Then we have $w(t) = K$ = constant at all time. You would think that the energy in such a pulse (treat at a voltage) would be infinite, sort of K^2 * infinite time. Yet is seems that $|W(\omega)|^2$ is integrable!

Resolution: In this case we have, from above,

$$Y(\omega) = (T_1)\ \text{sinc}(\omega T_1/2)\ \Sigma_{n=-\infty}^{\infty} e^{-in\omega T_1}$$

$$= (T_1)\ \text{sinc}(\omega T_1/2) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad -\infty < \omega < \infty$$

Although this is tamed a bit by the sinc function, it still contains δ functions and they contain infinite energy, so $W(\omega) = (1/T_1)X_{\mathbf{pulse}}(\omega)\ Y'(\omega)$ still has infinite energy due to these δ spikes.

Now if we consider only finite pulse trains, then $Y(\omega) = (T_1)\ \mathrm{sinc}(\omega T_1/2)\ \Sigma_n\ y_n\ e^{-in\omega T_1}$ has no delta spikes, and the sinc function really does tame it to a certain bandwidth. In this case, we expect finite energy, and that then agrees with a finite set of pulses.