This is some intermediate rewrite of DFT subsection (c). It no longer has this exact form because it was rewritten again later.

(c) The Discrete Fourier Transform for an Arbitrary Pulse

Recall from Fig 14.2 that we could consider $x_{pulse}(t)$ to be one which slopped over the boundaries of an interval of width T_1, such as the Gaussian shown there. Alternatively, we could consider $x_{pulse}(t)$ to be the pulse which is x(t) in the range $(0,T_1)$, as shown by the dark curve in Fig 14.2. In this section, we shall restrict $x_{pulse}(t)$ to be of the latter type, so $x_{pulse}(t)$ is non-zero only on the interval $(0,T_1)$, and in that region $x_{pulse}(t) = x(t)$. For such an $x_{pulse}(t)$, the infinite sum shown in (27.9) can be replaced with $\Sigma_{n=0}^{N-1}$, since $x_{pulse}(t) = 0$ outside this range. Our transform pair (27.9) and (27.11) become

$$c'_m \equiv (1/N) \sum_{n=0}^{N-1} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \qquad \text{projection = transform} \qquad (27.16)$$

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \; . \qquad \text{expansion = inverse transform} \qquad \textit{(27.11)} \qquad (27.17)$$

This is the official **Discrete Fourier Transform**. Due to the periodicity property (27.10), the sum in (27.17) could be taken over any set of N adjacent steps, and without much loss of generality we take these N steps to be 0,1,...N-1. If one were to regard the pulse as being translated to some other set of N steps like n = -3,-2,-1,0,1,... N-4, the coefficients c'_m would be exactly the same apart from a simple m-dependent phase. For example, let x'_{pulse} be the translated pulse. Then

$$d'_m \equiv (1/N) \sum_{n=-3}^{N-4} x'_{pulse}(t_n)\, e^{-imn(2\pi/N)} = (1/N) \sum_{n=-3}^{N-4} x_{pulse}(t_{n-3})\, e^{-imn(2\pi/N)}$$

$$= (1/N) \sum_{n'=0}^{N-1} x_{pulse}(t_{n'})\, e^{-im(n'-3)(2\pi/N)} \qquad // \; n' = n+3$$

$$= e^{+i3m(2\pi/N)} \{ (1/N) \sum_{n'=0}^{N-1} x_{pulse}(t_{n'})\, e^{-imn(2\pi/N)} \} = e^{+i3m(2\pi/N)}\, c'_m$$

$$= e^{+i3(m\omega_1)\Delta t}\, c'_m \qquad (27.19)$$

This result a reflection in the current context of the time-shift rule (12.1) which we restate here as

$$x(t + 3\Delta t) \quad \leftrightarrow \quad e^{+i3\omega\Delta t}\, X(\omega) \qquad \textit{(12.1)}$$

Note that c'_m refers to the spectral frequency $m\omega_1$.

We now summarize the DFT in a box:

Discrete Fourier Transform for an Arbitrary Pulse (27.20)

1. Let $x_{pulse}(t)$ be an arbitrary reasonable pulse defined for t in $(0,T_1)$.

2. Break up T_1 into N steps of width $\Delta t = T_1/N$. These relationships hold

$$\Delta t = T_1/N \qquad \omega_1 \equiv (2\pi/T_1) = 2\pi/(N\Delta t)$$
$$T_1 = N\,\Delta t \qquad (2\pi/N) = \omega_1 \Delta t$$
$$t_n = n\Delta t \qquad \omega_1 t_n = n\,\omega_1 \Delta t = n\,(2\pi/N) \qquad (27.3)$$

Thus, the sequence values of interest are $x_{pulse}(t_n)$ for n = 0,1,2...N-1.

3. Define the Discrete Fourier coefficients c'_m by this projection = transform:

$$c'_m \equiv (1/N) \sum_{n=0}^{N-1} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \qquad m = 0,1...N-1 \qquad (27.17)$$

4. The accompanying expansion = inverse transform is given by

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \qquad n = 0,1,...N-1 \qquad (27.18)$$

Comment: Just as with the Fourier Integral Transform, there several different conventions for stating the Discrete Fourier Transform. One could add any factor A in (17.17) and 1/A in (27.18). For example, with A = N, one moves the 1/N factor from one equation to the other. Another convention is the sign of the phase in the two equations.