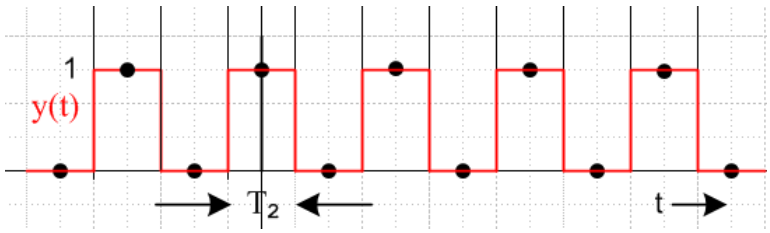
Old Example 3 from Section 25 PhL 3.28.13

I think this example adds little to my document and includes a bad error and just wastes pages anyway, so lets cut it and paste it here for safekeeping in case I want to bring it back to life.

Example 3: An infinite pulse train

Consider this infinite sequence of y_n samples, where we use T_2 in place of the usual T_1 ,



We can regard the red outline curve as an amplitude modulated pulse train whose pulse shape is a box of height A = 1 and width $\tau = T_2$. The Fourier Integral Transform spectrum of this box from (9.2) is

$$X_{pulse}(\omega) = T_2\,\mathrm{sinc}(\omega T_2/2) = T_2\,\mathrm{sinc}(\pi \frac{\omega}{\omega_2}) \qquad \omega_2 = 2\pi/T_2 \qquad (9.2)$$

From (25.4) the Fourier Integral Transform spectrum X(ω) of the pulse train is given by

$$X(\omega) = (1/T_2)\, X_{pulse}(\omega)\; Y'(\omega) \qquad (25.4)$$

where from (23.5),

$$Y'(\omega) = T_2 \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_2} = T_2 \sum_{n\ \mathrm{even}} e^{-in\omega T_2}$$

Using n = 2s, the sum on the right may be written $\Sigma_{s=-\infty}^{\infty}\, e^{-i2s\omega T_2}$. Then replace s with n to get,

$$Y'(\omega) = T_2 \sum_{n=-\infty}^{\infty} e^{-in2\omega T_2}$$ // probably wrong since we now really have N/2 limits

According to (13.2),

$$\sum_{n=-\infty}^{\infty} e^{\pm ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

Setting $k = 2\omega T_2$ we can write , using $\omega_2 \equiv 2\pi/T_2$,

$$2\pi\delta(k - 2\pi m) = 2\pi\delta(2\omega T_2 - 2\pi m) = \omega_2\delta(2\omega - m\omega_2) = (\omega_2/2)\delta(\omega - m\omega_2/2)$$

so then (13.2) with $k = 2\omega T_2$ says

$$\sum_{n=-\infty}^{\infty} e^{-in2\omega T_2} = (\omega_2/2) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_2/2)$$

and then

$$Y'(\omega) = T_2(\omega_2/2) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_2/2) = \pi \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_2/2) \qquad (25.10)$$

The Fourier Integral Spectrum of the pulse train from (25.4) is then

$$X(\omega) = (1/T_2)\,[X_{pulse}(\omega)]\ \{\, Y'(\omega)\,\}$$

$$= (1/T_2)\,[T_2\,\mathrm{sinc}(\pi\frac{\omega}{\omega_2})\,]\ \{\,\pi \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_2/2)\,\}$$

$$= \pi\,\mathrm{sinc}(\pi\frac{\omega}{\omega_2}) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_2/2) = \pi \sum_{m=-\infty}^{\infty} \mathrm{sinc}(m\frac{\pi}{2})\ \delta(\omega - m\omega_2/2)$$

$$= \pi\delta(\omega) + \pi \sum_{m=\pm\mathrm{odd}} \mathrm{sinc}(m\frac{\pi}{2})\delta(\omega - m\omega_2/2)$$

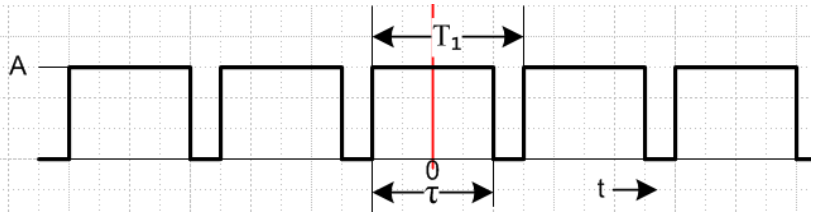
$$= (1/2)\,2\pi\delta(\omega) + \sum_{m=1\ 3\ 5}^{\infty} \mathrm{sinc}(m\frac{\pi}{2})\,2\pi\delta(\omega - m\omega_2/2) \qquad (25.11)$$

where we reflect the negative values to the positive side. The first term arises from the DC component of the waveform which is just (1/2), as (8.1) confirms.

We can regard the delta function lines at $\omega = m\omega_2/2$ shown in (25.11) as the image spectra of $Y'(\omega)$, and as usual they go out to infinity in both directions. These image spectra lines then propagate into $X(\omega)$ though they get tamped down by the sinc function at large ω (the even lines for $m \neq 0$ are completely removed by the sinc).

Comment: The energy associated with just a single one of these many delta lines is infinite (as shown in Appendix A (f)) so a frequency domain assessment of the pulse train energy is infinite. This is in agreement with the time domain assessment which is also infinite because there are an infinite number of pulses each of which carries a finite amount of energy.

We already examined this same pulse train from a different point of view in Section 17. There, we regarded it as *simple* pulse train of repeated pulses as shown in Fig ***



where $\tau = T_2 = T_1/2$, which means $\omega_2/2 = \omega_1$. The spectrum we found in Section 17 was this

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\, 2\pi\delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m\, 2\pi\delta(\omega - m\omega_1) \qquad \textit{(14.11)} \qquad \textit{(17.4)}$$

$$c_m = (1/T_1)\, X_{pulse}(m\omega_1) = (1\ T_2/T_1)\, \mathrm{sinc}(m\pi T_2/T_1) \qquad \textit{(14.7) and (14.2)} \qquad \textit{(17.5)}$$

$$= (1/2)\, \mathrm{sinc}(m\pi/2)$$

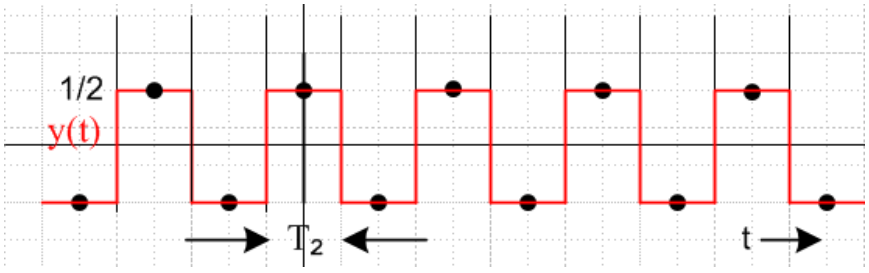
which says

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m\, 2\pi\delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} (1/2)\, \mathrm{sinc}(m\pi/2)\, 2\pi\delta(\omega - m\omega_1)$$

$$= (1/2)\, 2\pi\delta(\omega) + \sum_{m=1}^{\infty} \mathrm{sinc}(m\pi/2)\, 2\pi\delta(\omega - m\omega_2/2)$$

and this agrees with our result (25.11) since the even-m terms vanish.

If we add a DC offset (-1/2) to our first Figure above, we then get



and the corresponding spectrum is (25.11) with the DC term eliminated,

$$X(\omega) = \sum_{m=1\,3\,5}^{\infty} \mathrm{sinc}(m\tfrac{\pi}{2})\, 2\pi\delta(\omega - m\omega_2/2) = \sum_{m=1\,3\,5}^{\infty} \mathrm{sinc}(m\tfrac{\pi}{2})\, 2\pi\delta(\omega - m\omega_1) \qquad (25.12)$$