Old work from Appendix A PhL

3.26.05

Consider for example the central peak. We want to show that

$$\int_{-a}^{a} dk \left\{ \lim_{N\to\infty} \frac{1}{2\pi} \frac{\sin[(N+1/2)k]}{\sin(k/2)} \right\} = 1 \quad \text{for arbitrarily small a} \tag{A.18}$$

We have already argued that {...} will have zero integral over any region away from k = 0, so the entire integral must come from the region immediately close to k = 0. In this neighborhood, we can make the approximation sin(k/2) = k/2, so our task is then to show that

$$\int_{-a}^{a} dk \left\{ \lim_{N\to\infty} \frac{1}{2\pi} \frac{\sin[(N+1/2)k]}{(k/2)} \right\} = 1 \quad \text{for arbitrarily small a} \tag{A.19}$$

But this expression is not too hard to evaluate. We find

$$\int_{-a}^{a} dk\, \delta_5(k,N) = \int_{-a}^{a} dk \left\{ \lim_{N\to\infty} \frac{1}{2\pi} \frac{\sin[(N+1/2)k]}{(k/2)} \right\}$$

$$= \lim_{N\to\infty} \left(\int_{-a}^{a} dk \frac{1}{2\pi} \frac{\sin[(N+1/2)k]}{(k/2)} \right)$$

$$= \lim_{N\to\infty} \left(\frac{2}{\pi} \mathrm{Si}[(N+1/2)a] \right)$$

$$= \frac{2}{\pi} \lim_{N\to\infty} \mathrm{Si}[(N+1/2)a] = \frac{2}{\pi}\frac{\pi}{2} = 1. \tag{A.20}$$

The function Si(x) is the Sine Integral function, and it has the property $\lim_{x\to\infty} \mathrm{Si}(x) = \pi/2$. Thus, for an arbitrarily small a > 0, the limit is as shown above. Here is the integral done in Maple,

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int((1/(2*Pi))*sin((N+1/2)*k)/(k/2), k = -a..a);
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$$2\frac{\mathrm{Si}\left(\frac{1}{2}(2N+1)a\right)}{\pi}$$

The same analysis may be applied for any of the delta peaks. Thus we have demonstrated that

$$\lim_{N\to\infty} \delta_5(k,N) = \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m) \tag{A.21}$$