Square of Delta Function PhL 3.10.13

Products of distributions are not well-defined, though some have worked on the idea.
Of one of the distributions is a normal function, then OK,

(f) More on meaning of δ(0)

This subsection is a sort of coda on the subject of δ(0) where we further attempt to justify the use of δ(0) even though δ(0) is formally undefined.

Consider a voltage pulse v(t) with a shape corresponding to one our delta function models. If this voltage is placed across a resistor of value R = 1, the energy in the pulse is given by

$$E = \int_{-\infty}^{\infty} dt\, v^2(t) \ . \qquad \text{energy = time integral of power}$$

If we take the limit v(t)→δ(t), what happens? One is tempted to say

$$E = \int_{-\infty}^{\infty} dt\, \delta^2(t) = \int_{-\infty}^{\infty} dt\, \delta(t)\, \delta(t) = \delta(0) \int_{-\infty}^{\infty} dt\, \delta(t) = \delta(0) * 1 = \delta(0) = \infty$$

and one concludes (correctly) that the energy in a delta function pulse is infinite and positive. There are several problems with the this analysis.

First, we have already seen that with different δ models, we can get δ(0) to be any number we want, including +∞. -∞ and 0.

Second, the distribution $\langle \delta_\xi^2, \varphi \rangle = \delta(\xi)\varphi(\xi)$ is not a sensible distribution since this linear functional maps into something which is undefined at ξ = 0. In general the product of two distributions (in the sense we use it here) is not defined in the realm of distribution theory, unless one or both distributions are regular functions. In that case we could have, for example,

$$\langle f\, \delta_\xi, \varphi \rangle = \langle \delta_\xi, f\varphi \rangle = f(\xi)\varphi(\xi) = \text{ well defined}$$

Some people have tried to incorporate products of singular distributions into distribution theory, but it is not clear how their results apply in our current context (see for example Colombeau 1990).

Note that there are other meanings of the product of two distributions. One is called a convolution product which is like f(g(x)) for functions, while the other involves multiple variables like $\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$. Neither of these products involves products of symbolic functions in the same variable such as δ(t)δ(t).

So, how might we compute the energy in a delta function pulse? The only reasonable thing to do is to back δ(t) off to one of its models and see what happens. For example, suppose we take $\delta_1(t,A)$ as stated in (A.2), which is the simple box model of height A and width 1/A. Then

$$E = \lim_{A\to\infty} \left\{ \int_{-\infty}^{\infty} dt\ [\delta_1(t,A)]^2 \right\}$$

Now $[\delta_1(t,A)]^2$ is a box of width 1/A and height A^2 so its area is A. Then

$$E = \lim_{A\to\infty} \{ A \} = +\infty \qquad \text{// recall } \delta(0) = +\infty$$

Suppose we take our deviant delta function model (A.4) $\delta_2(k,A)$ shown in Fig ***. We get

$$E = \lim_{A\to\infty} \{ \int_{-\infty}^{\infty} dt \ [\delta_2(t,A)]^2 \}$$

Since the pulse is squared, the area under $[\delta_2(t,A)]^2$ is 3A and we then get

$$E = \lim_{A\to\infty} \{ 3A \} = +\infty \qquad \text{// recall } \delta(0) = -\infty$$

Finally, for the our second deviant model (A.5) also shown in Fig *** we get

$$E = \lim_{A\to\infty} \{ A/2 \} = +\infty \qquad \text{// recall } \delta(0) = 0$$

Thus, we get $E = +\infty$ regardless of the value of $\delta(0)$ in the model.

Despite this discussion which shows that $\delta(0)$ is undefined, we nevertheless use $\delta(0)$ with a particular model in mind because it allows us to avoid dealing with specific boundaries in equations.

In Example 2 of the previous subsection, we saw the use of $2\pi\delta(0)$ as meaning 2N+1 for a very long pulse train starting at -N in the distant past and ending at +N in far future. We would rather think in terms of an infinite pulse train and $2\pi\delta(0)$, but the physical meaning is a very long pulse train and 2N+1. It is just a convenient notation.

Another common example has to do with "box normalization" which in one dimension is the model presented as (A.13),

$$\int_{-L/2}^{L/2} dx \ e^{ikx} = 2\pi \ \frac{\sin(kL/2)}{\pi k} = 2\pi \ \delta_4(k,L/2)$$

$$\int_{-L/2}^{L/2} dx \ e^{i0x} = L = 2\pi\delta(0)$$

In this case, we use $2\pi\delta(0)$ to represent the length of a box which is some very large L. Rather than carry the finite L along in all equations, we can use $2\pi\delta(0)$ as needed represent this long length.

The conclusion here is that we can use $2\pi\delta(0)$ as a notational device provided we are very careful as to the meaning of that device. We must always know how to undo the limit, and that implies a specific delta function model for a specific application.