Additions and Verifications from my Library Bike Voyage PhL 5.11.13

On May 11 I rode to Marriott and perused a few books related to spectral doc. The Xiong book, which I later downloaded, revealed that α and β are related to μ and σ^2. This is in now in my Appendix D and is incorporated into spectral doc in a reasonable way. I was too stupid to even realize that $\langle y_m y_n \rangle = \langle y_m \rangle \langle y_n \rangle$, and I also had the meaning of correlation wrong. It is all cleared up now I think.

1. Everybody uses the phrase "power spectral density" PSD, so probably I should change my phrase to match or at least add a comment on same.

2. My "general" pulse train is really just a PAM digital signal, I should mention that somewhere. I might note that B&D allow for two different pulse shapes, whereas I only allow for one with two different amplitudes. We treated different problems, perhaps a comment on this.

2. Xiong's appendix gets my α and β formula, but interprets the parameters in a way that seems might be good to add, since I already have a correlation theory section especially. Here is my formula

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi \delta(\omega T_1 - 2\pi m) \} \qquad (35.11)$$

$n \neq m \qquad \alpha \equiv \langle y_m y_n \rangle = [pp]\,AA + [p(1-p)]\,AB + [(1-p)p]BA + [(1-p)(1-p)]BB$

$n = m \qquad \beta \equiv \langle y_m^2 \rangle = [p]AA + [(1-p)]\,BB\,. \qquad (35.6)$

I am confused by the X appendix A. He gets this result

$$= |G(f)|^2 \lim_{N\to\infty} \frac{1}{(2N+1)T} \sum_{n=-N}^{N} \sum_{m=-N}^{N} E\{a_n a_m\} e^{j(m-n)\omega T}$$

where I like $E\{a_n a_m\}$. He then has

where $k = n - m$, $E\{a_n a_m\} = E\{a_n a_{n-k}\} = R(k)$ is the autocorrelation function of the data bits. Note that $R(-k) = R(k)$. Realizing the inner summations become

and yes, this is a sort of digital autocorrelation function. But then I think basically he is claiming that

$\langle a_n a_m \rangle = \langle a_n \rangle \langle a_m \rangle$

in the uncorrelated case. Well I guess that is right, I overlooked the obvious. Suppose I add to my list

$\mu = \langle y_m \rangle = [p]A + [p-1]B$

$\alpha \equiv \langle y_m y_n \rangle = \langle y_m \rangle \langle y_n \rangle = \mu^2$ when $m \neq n$

This never appeared in my Ng notes because I guess it was not relevant. I might add this

$$\text{corr}(X,Y) = \int\int x\, y\, p(x,y)\, dx\, dy \qquad \text{// general case}$$

When things are uncorrelated, the idea is that

$$p(x,y) = p(x)p(y)$$

so we get

$$\text{corr}(X,Y) = \int\int x\, y\, p(x,y)\, dx\, dy = \left(\int x\, p(x)\, dx\right)\left(\int y\, p(y)\, dy\right) = \mu_x \mu_y$$

In the discrete world this would be

$$p(x_1,x_2) = \Sigma_{i=1}^{m} P(\mathbf{x}^{(i)})\, \delta(x_1 - x^{(i)}_1)\, \delta(x_2 - x^{(i)}_2) = \Sigma_{i=1}^{m} P(\mathbf{x}^{(i)}) \delta^{(2)}(\mathbf{x} - \mathbf{x}^{(i)})$$

If the two variables are independent, then [i labels my little clot points]

$$P(\mathbf{x}^{(i)}) = P(x_1^{(i)})\, P(x_2^{(i)})$$

then

$$p(x_1,x_2) = \Sigma_{i=1}^{m} P(x_1^{(i)})\, P(x_2^{(i)})\, \delta(x_1 - x^{(i)}_1)\, \delta(x_2 - x^{(i)}_2)$$

$$= \Sigma_{i=1}^{m} [P(x_1^{(i)})\, \delta(x_1 - x^{(i)}_1)][P(x_2^{(i)})\, \delta(x_2 - x^{(i)}_2)]$$

or

$$p(x,y) = \Sigma_{i=1}^{m} [P(x^{(i)})\, \delta(x - x^{(i)})][P(y^{(i)})\, \delta(y - y^{(i)})]$$

Then for example,

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, x\, y\, p(x,y)\,.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, x\, y\, \Sigma_{i=1}^{m} [P(x^{(i)})\, \delta(x - x^{(i)})][P(y^{(i)})\, \delta(y - y^{(i)})]$$

$$= \Sigma_{i=1}^{m} \int_{-\infty}^{\infty} dx\, x[P(x^{(i)})\, \delta(x - x^{(i)})] \int_{-\infty}^{\infty} dy\, y[P(y^{(i)})\, \delta(y - y^{(i)})]$$

$$= \Sigma_{i=1}^{m} \{ P(x^{(i)})\, x^{(i)} \} \{ P(y^{(i)}\, y^{(i)} \}$$

Maybe do it instead this way

$$E(XY) = \Sigma_i \Sigma_j \, x_i y_j \, p(x_i, y_j) = \Sigma_i \Sigma_j \, x_i y_j \, p(x_i) q(y_j) = [\Sigma_i x_i \, p(x_i)] \, [\Sigma_j y_j \, q(y_j)] = E(X)E(Y)$$

OK, but then how do I get something with a single sum

$$E(XY) = (1/I) \, \Sigma_i \, x_i y_i$$

How can I reconcile these two expressions for E(XY) ?

Idea: I think I am using label i to mean two different things:

- i is a label of a pulse train in the ensemble = label of training point in Ng world
- i is used to label discrete values of a variable as in $x_i = \{1, 2, 3\}$

I am confusing these two meanings and getting meaningless results.

The ensemble pulse trains and the training points all have the same weight in whatever we compute, so they are all equally as likely in some vague sense.

OK, let's not dwell on this Ng connection right now. Go back to

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \; \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \; \} \, . \qquad (35.7)$$

I can write this setting $\alpha = \mu^2$, I now accept that fact, so then

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \; \mu^2 \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \; \} \, . \qquad (35.7)$$

Now what about the other one?

$$\beta \equiv \langle y_m^2 \rangle$$

Recall that

$$\text{variance} \equiv \langle [y_m - \langle y_m \rangle]^2 \rangle = \int_{-\infty}^{\infty} (x-\mu_x)^2 \, p(x) \, dx = \sigma^2$$

So I guess I can write

$$\sigma^2 = \langle y_m^2 \rangle - 2 \, \langle y_m \rangle^2 + \langle y_m \rangle^2 = \langle y_m^2 \rangle - \langle y_m \rangle^2 = \text{variance} = \beta - \mu^2.$$

Then I can write

$$\beta = \sigma^2 + \mu^2 \text{ and then}$$

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \; \mu^2 \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + (\sigma^2+\mu^2) \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \; \} \, . \qquad (35.7)$$

OK, now jump to my endgame where I have for uncorrelated case

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \, \{ \, (\beta-\alpha) \; + \alpha \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) \, \} \qquad (35.11)$$

Then since I just showed that

$$\alpha \equiv \langle y_m y_n \rangle = \langle y_m \rangle \langle y_n \rangle = \mu^2$$
$$\beta = \sigma^2 + \mu^2$$
$$\beta - \alpha = \sigma^2$$

I could write my result as

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \, \{ \, (\sigma^2 \; + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) \, \} \qquad (35.11)$$

But using these new names does not really simplify any of my calculations.

OK, I have my little probability digression sitting in exactly the right place to add something on this subject with minimal disruption. But I need a way to talk about independence in the discrete world.

$$\langle y_m y_n \rangle = E(Y_m Y_n) = (1/I) \sum_{i=1}^{I} y_m^{(i)} y_n^{(i)} \equiv \text{corr}(Y_m Y_n)$$

I think I have to stop and restart this subject. I got a lot done but did not solve the problem.