27. The Discrete Fourier Transform

This entire section is on probation for a while.

First, we consider the Discrete Fourier Transform

Fourier Integral Transform (both sides are integrals)

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

Digital Fourier Transform (spectrum is a sum, expansion is an integral)

$$X'(\omega) \equiv \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega t_n} \qquad \text{projection = transform} \qquad (22.2)$$

$$x(t_n) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega t_n} \qquad \text{expansion = inversion} \qquad (22.4)$$

Is this a pulse or a pulse train? Question: Suppose we apply 22.2 to signal which happens to be finite. Look at the derivation of the above and what then happens? Something is fishy!

Here is a web site that is not to clean , but here is one conclusion

http://fourier.eng.hmc.edu/e101/lectures/handout4/node3.html

II	Discrete (t_0), Non-periodic	Periodic ($F = 1/t_0$), Continuous
	$x[m] = \int_F X_F(f) e^{j2\pi f m t_0} df/F$	
	$x[m] = \int_F X_F(f) e^{j2\pi f m t_0} df/F$	$X_F(f) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f m t_0}$

Here you see discrete time x[m] given as a one-period integral, as in mine, and you see the spectrum as a sum just as in mine. So this is my Digital Fourier Transform. The time domain is discrete steps, but the ω domain is a sum of continuous functions, so continuous. OK.

When I do my Digital FT Section 22, I never mention a pulse train and I never mention X_{pulse}.

(a) Discrete Fourier Transform of a Finite Pulse x(t)

I conjecture that this is the correct transform

$$x(t_n) = \sum_{m=0}^{N-1} c'_m \, e^{+imn(2\pi/N)} \qquad n = 0,1....N-1$$

$$c'_m \equiv (1/T_1) \sum_{n=0}^{N-1} \Delta t \, x(t_n) \, e^{-im\omega_1 t_n} \qquad m = 0,1,....N-1$$

$$\Delta t = T_1/N \qquad T_1 = N\,\Delta t \qquad \omega_1 = (2\pi/T_1) = 2\pi/(N\Delta t) \qquad t_n = n\Delta t \qquad (27.3)$$

The entire pulse is assumed to fit inside the interval $(0,T_1)$. In this case,

t_n lies in $(0,n_{max}\Delta t) = (0,NT_1/N) = (0,T_1\ )$ check

So index n runs from n = 0 to n = N-1 I suspect.

Now let's verify this transform as in spectral doc:

$$x(t_n) = \sum_{m=0}^{N-1} c'_m \, e^{+imn(2\pi/N)} = \sum_{m=0}^{N-1} [(1/N) \sum_{k=0}^{N-1} x(t_k) \, e^{-imk(2\pi/N)}] \, e^{+imn(2\pi/N)}$$

$$= (1/N) \sum_{k=0}^{N-1} x(t_k) \sum_{m=0}^{N-1} e^{+im(2\pi/N)(n-k)} \qquad (27.12)$$

$$= (1/N) \sum_{k=0}^{N-1} x(t_k) \; N \sum_{m=-\infty}^{\infty} \delta_{k,n-mN} = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} x(t_k) \, \delta_{k,n-mN}$$

$$= x(t_{n-mN}) = x(t_n - mT_1) \qquad . \qquad (27.14)$$

$$= 1/N) \sum_{k=0}^{N-1} x(t_k) \; N \sum_{m=-\infty}^{\infty} \delta_{k,n-mN}$$

$$= (1/N) \sum_{k=0}^{N-1} x(t_k) \; N \, \delta_{k,n} = \quad x(t_n)$$

(a) Discrete Fourier Transform of a Finite Pulse x(t) of width T_1

Recall the