Inverting the Hilbert Transform PhL 4.3.13

Very early notes on this subject, I think all has been absorbed into Appendix C (h).

$$X_h(\omega) \equiv -(1/\pi) \fint_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega}$$

$$x_h(t) = i\, \mathrm{sgn}(t)\, x(t)$$

This is a diagonalized convolution equation in t space. Recall our inverse theorem

$$A(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, B(\omega-\omega')C(\omega') \quad \Leftrightarrow \quad a(t) = b(t)\, c(t) \tag{3.7}$$

$$X_h(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, \mathrm{SGN}(\omega-\omega')i\, X(\omega') \quad \Leftrightarrow \quad x_h(t) = i\, \mathrm{sgn}(t)\, x(t) \tag{3.7}$$

Now what is the FT of sgn(t)?

$$\mathrm{SGN}(\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{sgn}(t)\, e^{-i\omega t} = -2i \int_{0}^{\infty} dt\, \sin(\omega t) = 2i/\omega \qquad \text{I think}$$

Then we get

Inverse Hilbert Transform: Note that

$$[\hat{G}(f)]^{\wedge} = (-\mathrm{j}\ \mathrm{sgn}(f))^2 G(f) = -\mathrm{sgn}^2(f)G(f).$$

Except at $f = 0$, $\mathrm{sgn}^2(f) = 1$. Thus, unless $G(f)$ has some sort of singularity (e.g., a delta function) at $f = 0$, we get $\mathcal{H}[\mathcal{H}[g(t)]] = -g(t)$. A delta function at $f = 0$ corresponds to a nonzero dc offset; which, as already remarked upon, is lost by the Hilbert transform. Thus, assuming that $g(t)$ has zero mean[1], we may recover $g(t)$ from $\hat{g}(t)$: the inverse Hilbert transform is given by applying the Hilbert transform again, and negating the result:

$$g(t) = -\mathcal{H}[\hat{g}(t)] = -\hat{g}(t) * \frac{1}{\pi t}.$$

In general, we have, for some constant c,

$$g(t) = -\hat{g}(t) * \frac{1}{\pi t} + c.$$

$$X_h(X_h(\omega) = -\omega \qquad \Rightarrow X_h^{-1}(\omega) = -X_h(\omega)$$