8. Some simple examples of spectra

(a) The spectrum of x(t) = 1 :

In this example, x(t) is a constant over all time. Using (1.1) and (2.1), we find:

$$x(t) = 1 \qquad X(\omega) = 2\pi\,\delta(\omega)\ . \tag{8.1}$$

This comes as no surprise. For a DC signal, all the energy is concentrated at zero frequency. Of course x(t) = 1 does not respect the requirement $\int_{-\infty}^{\infty} dt\,|x(t)| < 0$, which is why the spectrum is a distribution.

(b) The spectrum of $x(t) = \delta(t - t_1)$:

Here x(t) is an infinitely narrow pulse of area 1, positioned at $t=t_1$. Using (1.1), we get the following Fourier spectrum:

$$x(t) = \delta(t - t_1) \qquad X(\omega) = e^{-i\omega t_1}\ . \tag{8.2}$$

The spectrum X(ω) has a constant magnitude 1 for all ω, out to infinite frequency. For such a pulse at t=0,

$$x(t) = \delta(t) \qquad X(\omega) = 1 \tag{8.3}$$

and here the phase is constant. This result is (8.1) with $\omega \leftrightarrow t$ and the constant adjusted due to the asymmetry of the transform in our adopted convention.

(c) The regular Fourier Integral Transform spectrum of the Heaviside step function θ(t) is the somewhat peculiar first line following, whereas the generalized Fourier Integral Transform gives the second line

$$x(t) = \theta(t) \qquad X(\omega) = \text{"}\frac{1}{i\omega}\text{"} = \frac{1}{i\omega+\varepsilon} = \frac{1}{i}\frac{1}{\omega - i\varepsilon} = \frac{1}{i}\,[\,pf(1/\omega) + i\pi\delta(\omega)] \tag{8.4}$$

$$X(\omega) = \frac{1}{i\omega} \qquad \text{// generalized Fourier Integral Transform of (6,4,5)}$$

which we now explain. This Heaviside x(t) is also not in the class of functions for which the Fourier Transform is defined ($\int_{-\infty}^{\infty} dt\,|x(t)| < 0$). We bring θ(t) into the acceptable class by replacing θ by θ_ε where,

$$\theta_\varepsilon(t) \equiv \begin{cases} e^{-\varepsilon t} & t > 0 \\ 0 & t < 0 \end{cases} \qquad \text{for some very small } \varepsilon > 0$$

Then

$$X_\varepsilon(\omega) = \int_{-\infty}^{\infty} dt\, \theta_\varepsilon(t)\, e^{-i\omega t} = \int_0^{\infty} dt\, e^{-\varepsilon t}\, e^{-i\omega t} = \frac{1}{i\omega+\varepsilon}$$

and then $X(\omega) = \lim_{\varepsilon\to 0} X_\varepsilon(\omega) = (1/i\omega)$. *But* we really have to think of $(1/i\omega)$ as meaning the limit of $\frac{1}{i\omega+\varepsilon}$. To see why, we now compute x(t) from the inversion formula,

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega \frac{1}{i\omega+\varepsilon}\, e^{+i\omega t} = (1/2\pi i) \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon}\, e^{+i\omega t} . \qquad \text{pole at } \omega = +i\varepsilon$$

For $t < 0$, close the ω contour down and get 0 since the great circle vanishes. For $t > 0$ close up and pick up the pole reside to get $x(t) = e^{-\varepsilon t}$. For $t = 0$, we let the pole move to the real axis from above and deflect the contour down

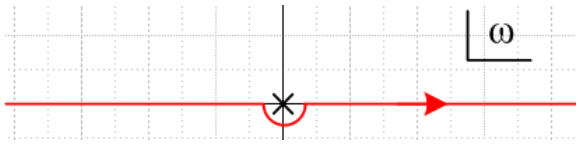


Fig 8.1

In the limit the contour is shrunk around the pole, the contributions from $(-\infty,0)$ and $(0,\infty)$ cancel. These two terms are known as a principle value integral and we have

$$PV \int_{-\infty}^{\infty} d\omega\, (1/\omega) = 0$$

since the left and right sides cancel (even range integral of an odd function). All that is left is the half turn around the pole which picks up half the residue at the pole (one can show) so the result is then

$$x(0) = (1/2\pi i) \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon} = (1/2\pi i)\ (1/2)\ (2\pi i * 1) = 1/2$$

and we obtain the fact that $\theta(0) = 1/2$ as was shown in Fig 1.1.

What we have just done is sometimes written using the following obscure notation (see for example Stakgold Chapter 1 page 50 (1.27))

$$\frac{1}{\omega - i\varepsilon} = pf(1/\omega) + i\pi\delta(\omega) \qquad \text{// similarly } \frac{1}{\omega + i\varepsilon} = pf(1/\omega) - i\pi\delta(\omega) \qquad (8.5)$$

where pf means pseudofunction and is a symbolic function like the delta function which acquires meaning when it is placed inside an integral. The meaning is that when $pf(1/\omega)$ is inside an integral, the integral is a principal value integral at $\omega = 0$, so

$$\int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon} = \int_{-\infty}^{\infty} d\omega\, pf(1/\omega) + \int_{-\infty}^{\infty} d\omega\, i\pi\delta(\omega)$$

$$= \text{PV} \int_{-\infty}^{\infty} d\omega\, (1/\omega) + i\pi \quad = 0 + i\pi \quad = i\pi$$

and then

$$x(0) = (1/2\pi) \int_{-\infty}^{\infty} d\omega \frac{1}{i\omega+\varepsilon} \quad = (1/2\pi i) \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon} \quad = (1/2\pi i)\, i\pi \quad = 1/2\ .$$

If we apply our upcoming differentiation rule (11.1) [which is to multiply by iω] we find that

$$\theta(t) \leftrightarrow \frac{1}{i\omega+\varepsilon} \qquad\qquad => \qquad\qquad \delta(t) = d\theta(t)/dt \quad \leftrightarrow \quad i\omega \frac{1}{i\omega+\varepsilon} = 1$$

which agrees with (8.3) above.

Using the *generalized* Fourier Integral Transform stated in (6.4) and (6.5), we can regard $X(\omega) = 1/(i\omega)$ without all the ε business since the ω recovery contour in (6.5) runs below all singularities in the ω plane, which contour, when deformed up, gives Fig 8.1 and all the results quoted above. Applying our Laplace equivalence notion (6.8), we would predict from $X(\omega) = 1/(i\omega)$ that

$$\mathcal{L}\{\theta(t), s\} = \mathcal{X}(s) = X(s/i) = \frac{1}{i(s/i)} \quad = \frac{1}{s}$$

which is in agreement with any Laplace table.