Section 31 d Version 5

(d) application to coaxial cable

Consider an infinitely long coaxial cable driven at its left end at z = 0. A coaxial cable acts as a filter G(ω). In the frequency domain, it we drive the cable with I(ω), the output O(ω,z) at z is

$$O(\omega, z) = G(\omega, z)\, I(\omega) \qquad G(\omega) = e^{-\gamma(\omega) z} \qquad \gamma(\omega) = \alpha(\omega) + i\beta(\omega) .$$

For a coaxial cable one has

$$\gamma = \sqrt{(R+i\omega L)(G+i\omega C)}$$

where R,L,G and C are resistance, inductance, conductance (across the dielectric) and capacitance all per unit length of the cable. If we ignore ohmic losses in both the conductor and the dielectric, then R = G = 0 and we get

$$\gamma = i\omega\sqrt{LC} .$$

The inductance L is generally independent of ω, but $C = \varepsilon(\omega)2\pi\varepsilon_0/\ln(b/a) = k_1\, \varepsilon(\omega)$, where ε(ω) is the dielectric "constant", which in general is not constant as a function of ω. Thus we have

$$\gamma(\omega) = i\omega\sqrt{Lk_1}\,\sqrt{\varepsilon(\omega)} .$$

From Maxwell's equations one knows that the index of refraction of a medium is given by

$$n(\omega) = \sqrt{\mu\varepsilon} = \sqrt{\varepsilon(\omega)}\ /k_2 .$$

So we then have,

$$\gamma(\omega) = i\omega\sqrt{Lk_1}\, k_2\, n(\omega) = i\omega k_3\, n(\omega)$$

$$= i\omega k_3[\, Re(n) + i\, Im(n)\,] = -\omega k_3 Im(n) + i\omega k_3 Re(n) = \alpha(\omega) + i\beta(\omega) .$$

so

$$\alpha(\omega) = -\omega k_3 Im[n(\omega)] \qquad \beta(\omega) = \omega k_3 Re[n(\omega)] .$$

If it were true that Re[n(ω)] were independent of ω, then β(ω) would have linear phase and we would then have constant group delay and no dispersion in the coaxial cable.

It turns out that the index n(ω) has the right properties for the dispersion relations (31.3) to be valid, so

$$Re[n(\omega)] = Re[n(\infty)] + (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{Im[n(\omega')]}{\omega'-\omega} \qquad \textit{(31.3a)}$$

$$\mathrm{Im}[n(\omega)] = \mathrm{Im}[n(\infty)] - (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}[n(\omega')]}{\omega'-\omega} \qquad (31.3b)$$

where usually $\mathrm{Re}[n(\infty)] = 1$ and $\mathrm{Im}[n(\infty)] = 0$. If it happened that $\mathrm{Im}[n(\omega)]$ were very small, then (31.3a) says that $\mathrm{Re}[n(\omega)] = \mathrm{Re}[n(\infty)] =$ constant, just what we want to get no cable dispersion.

In a non-polar dielectric, like polyethylene or teflon, $n(\omega)$ does in fact have a very small imaginary part for frequencies below the electromagnetic resonances of the medium. Thus, if we could ignore ohmic losses in the conductors, coaxial cables using these materials as dielectrics would be non-dispersive up to infrared frequencies -- where vibrational and rotational resonances set in.

Dispersion relations are often written for other functions such as the dielectric constant $\varepsilon(\omega)$ itself. In this case one can regard the relationship between electric displacement D and electric field E

$$D(\omega) = e(\omega)\, E(\omega)$$

as a "filter", where everything is evaluated at the same point in space.