Correlation Stuff PhL 3.22.13

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I have now added a version of this text block near the start of Section 35.

Expectation, correlation and covariance. In the field of statistics, one can imagine a "random variable" A_m associated with position m in the pulse train, and this variable can take certain values a_k (perhaps 0 and 1). If all pulse trains in our ensemble of pulse trains count the same in our average (they do), then one can write $\langle a_m a_n \rangle$ as

$$E(A_m A_n) = (1/I) \sum_{i=1}^{I} a^i_m a^i_n \qquad \text{in the sense of} \qquad E(XY) = (1/I) \sum_{i=1}^{I} x^i y^i$$

where E(q) means the expectation value of quantity q. This particular expectation value has a special name: it is called the correlation of X and Y. So in fact we have

$$\langle a_m a_n \rangle = E(A_m A_n) = (1/I) \sum_{i=1}^{I} a^i_m a^i_n \equiv \text{corr}(A_m A_n) \qquad \text{// correlation of } A_m \text{ and } A_n$$

Two other expectation values of interest are these, the mean and the covariance,

$$E(A_m) = (1/I) \sum_{i=1}^{I} a^i_m = \langle a_m \rangle \qquad \text{// mean of } A_m$$

$$E([A_m - \langle a_m \rangle][[A_n - \langle a_n \rangle]) = (1/I) \sum_{i=1}^{I} [a^i_m - \langle a_m \rangle] [a^i_n - \langle a_n \rangle]$$

$$= \text{cov}(A_m A_n) \qquad \text{// covariance of } A_m \text{ and } A_n$$

If it happens that $\langle a_m \rangle = 0$ for a certain pulse train, then covariance and correlation are the same.

The following path did not seem to work out, it is replaced with cross-correlation discussion in Spectral. But I will keep it alive here.

Autocorrelation. Consider again this equation from above,

$$E(XY) = (1/I) \sum_{i=1}^{I} x^i y^i = \text{corr}(XY).$$

Suppose there are I = 2N+1 pulse trains in our ensemble, and we choose to enumerate them with index i in the range (-N,N) so then

$$E(XY) = \frac{1}{(2N+1)} \sum_{i=-N}^{N} x^i y^i = \text{corr}(XY).$$

We are summing over a set of (2N+1) random "events" (pulse trains in example above). Random variable X takes value x^i in the i^{th} random event. If x can take a *continuous* set of values, then we can enumerate them by some continuous parameter s which by fiat we require to lie in a range (-B,B). Now we have a function x(s). Then think of $x^i = x(s_i)$. Divide the s parameter space into (2N+1) pieces of width Δs so that $s_i = i\Delta s$ where $\Delta s = (2B)/(2N+1)$. The above then becomes

$$E(XY) = \frac{1}{2B} \sum_{i=-N}^{N} \Delta s\, x(i\,\Delta s)\, y(i\,\Delta s) = \text{corr}(XY).$$

Now let $N \to \infty$ so $\Delta s \to 0$ and we get

$$E(XY) = \frac{1}{2A} \int_{-A}^{A} ds\, x(s)\, y(s) = \text{corr}(XY).$$

Now define $\mathcal{E}_A(XY) \equiv 2A\, E(XY)$ and $corr_A(XY) = 2A\, \text{corr}(XY)$ to get

$$\mathcal{E}_A(XY) = \int_{-A}^{A} ds\, x(s)\, y(s) = corr_A(XY)$$

Now take the limit $A \to \infty$ to get

$$\mathcal{E}(XY) = \int_{-\infty}^{\infty} ds\, x(s)\, y(s) = corr(XY)$$

where $\mathcal{E}(XY) \equiv \mathcal{E}_\infty(XY)$ and the same for $corr(XY)$.

What does this mean? We have an infinite number of independent random events labeled by continuous variable s (instead of discrete i). In event s, random variable X takes some value x(s) and random variable Y takes value y(s). The integral is then the correlation between X and Y.

Now imagine that the value $x^i = x(t_i)$ where x(t) is a continuous function. And let $\Delta t \equiv (2N+1)^{-1}$. Then

$$E(XY) = \sum_{i=-N}^{N} \Delta t\, x(t_i) y(t_i) = \text{corr}(XY).$$

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Suppose x^i could take a continuous set of values in range (-N,N), so we would then call it x(t). Then ignoring the factor (1/I) we might write the above as

$$E(XY) = \frac{1}{(2N+1)} \int_{-N}^{I/2} dt\, x(t)y(t) = corr(XY).$$

Letting I go to infinity, we could write this as (see Appendix A regarding $\delta(0)$),

$$[2\pi\delta(0)]\, E(XY) = \int_{-\infty}^{\infty} dt\, x(t)y(t) = [2\pi\delta(0)]corr(XY).$$

We could then redefine our notation so this reads

$$\mathcal{E}(XY) = \int_{-\infty}^{\infty} dt\, x(t)y(t) = corr(XY)$$

So this integral gives the correlation between X which takes values x(t) and Y which takes values y(t). Now suppose $y(t) = x(t+a) = x_{\mathbf{a}}(t)$. Then we have

$$\mathcal{E}(XX_{\mathbf{a}}) = \int_{-\infty}^{\infty} dt\, x(t)\, x_{\mathbf{a}}(t) = corr(XX_{\mathbf{a}}) = \int_{-\infty}^{\infty} dt\, x(t)\, x(t+a)$$

This then is the correlation between x(t) and a time-shifted copy of itself. Since the same function appears twice in the integral, this integral is called the *auto*-correlation function of x(t) associated with a shift by amount a. Looking back now at (32.1) and changing its variable

$$r_{\mathbf{x}}(a) \equiv \int_{-\infty}^{\infty} dt\, x(t)\, x(t+a) \qquad (32.1)$$

we see that our autocorrelation function $r_{\mathbf{x}}(a) = \mathcal{E}(XX_{\mathbf{a}}) = corr(XX_{\mathbf{a}})$.