Repeat Statistical Stuff with Finite N everywhere. PhL 3.26.05

14. The Spectrum of a Pulse Train

A pulse train consists of a set of pulses of shape $x_{pulse}(t)$ separated by time T_1. We assume that $x_{pulse}(t)$ is some "reasonable" (non-pathological) function.

$$x(t) = \sum_{n=-N}^{N} x_{pulse}(t - t_n) \qquad t_n = nT_1 \qquad \text{pulse train} \qquad (14.1)$$

Let $X_{pulse}(\omega)$ be the spectrum of $x_{pulse}(t)$. This $x_{pulse}(t)$ does not really have to be a "pulse", but it is convenient to think of it as such. We imagine that $x_{pulse}(t)$ is a function that is somewhat localized in the region of t=0, and vanishes for very large positive and negative time. The half-width of the pulse can be larger than T_1 as discussed below, so pulses can overlap. Thus, our $x_{pulse}(t)$ is not itself periodic, and we thus expect *it* to have a continuous spectrum.

For example, from (9.2) we already know the spectrum of a single square wave pulse of height A and width τ centered at t=0:

$$X_{pulse}(\omega) = (A\tau)\ \mathrm{sinc}(\omega\tau/2)\ . \qquad (14.2)$$

Now consider a second pulse which is a copy of our original pulse, but which is translated T_1 units to the right in time. From (12.1), we know the spectrum of this second pulse:

$$X(\omega, \text{second pulse}) = X_{pulse}(\omega)\ e^{-i\omega T_1}\ .$$

Now construct the infinite periodic wave by superposing pulses at $t = 0, \pm T_1, \pm 2T_1, \ldots$. We get,

$$X(\omega) = X_{pulse}(\omega) \sum_{n=-N}^{N} e^{i\omega n T_1} \qquad (14.3)$$

We quote now a result from Appendix A

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} = 2\pi\, \delta_5(k,N) \qquad \textit{(A.25)}$$

Thus (14.3) becomes, setting $k = \omega T_1$

$$X(\omega) = X_{pulse}(\omega)\ 2\pi\ \delta_5(\omega T_1, N) \qquad (14.4)$$

STOP HERE and go to (3.31) below and do edits there

In getting to the next step, we use the property of delta functions $\delta(ax) = \delta(x)/a$, and we define $\omega_1 \equiv 2\pi/T_1$ = the fundamental angular frequency of our periodic waveform. The result is: red flag?

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} \delta\left(\frac{\omega}{\omega_1} - m\right) . \qquad (14.5)$$

Notice that the delta function argument is dimensionless. This is a nice form in theory, but in practice we want ω inside the delta function to be isolated so that, when (14.5) is inserted into a dω integration, we don't pick up any new constants. Thus, we extract the $\omega_1 = 2\pi/T_1$ factor, keeping the 2π part right with δ():

$$X(\omega) = \sum_{m=-\infty}^{\infty} (1/T_1)X_{\mathbf{pulse}}(\omega)\ 2\pi\delta(\omega - m\omega_1) . \qquad (14.6)$$

Thus, we have a set of evenly spaced delta function spikes which occur at these frequencies:

$$\omega_m = m\omega_1 \quad m = 0,1,2,3..... \qquad (14.7)$$

In formulas like (14.6) , one should keep in mind that the variable ω appearing inside $X_{\mathbf{pulse}}(\omega)$ can at any time be replaced with $\omega_m \equiv m\omega_1$ due to the presence of the delta function. In general, one has,

$$f(\omega)\ \delta(\omega - a) = f(a)\ \delta(\omega - a) .$$

Both sides of this equation are zero when $\omega \neq a$, and at $\omega = a$, $f(a) = f(\omega)$.

Because the quantity $(1/T_1)X_{\mathbf{pulse}}(\omega)$ is going to occur very frequently in the following discussion, we shall now define a more compact notation for it as follows:

$$c(\omega) \equiv (1/T_1)X_{\mathbf{pulse}}(\omega) . \qquad (14.8)$$

Thus, c(ω) is nothing more than our (continuous) pulse spectrum divided by the fundamental period T_1. We can then rewrite (14.6) as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\ \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c(\omega_m)\ 2\pi\ \delta(\omega - m\omega_1) . \qquad (14.9)$$

As was just noted above, we can harmlessly replace ω with $\omega_m = m\omega_1$ inside c(ω) in (14.8). This leads us to define a set of numbers as follows

$$c_m \equiv c(\omega_m) = c(m\omega_1) \qquad (14.10)$$

These numbers are just the values that the function c(ω) takes at our delta spike frequencies. We arrive then at our final form for the spectrum of a pulse train,

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1) . \qquad (14.11)$$

Now we are ready to summarize all these results:

Fourier Integral Transform of a Pulse Train (14.12)

1. Let $x_{pulse}(t)$ be any reasonable pulse. Construct a pulse train x(t) with spacing T_1:

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) . \qquad (14.1)$$

By its construction, x(t) is periodic with period T_1, which we can write formally as:

$$x(t + nT_1) = x(t) . \qquad n = \text{any integer}$$

If x(t) is a *known* periodic function of period T_1, a candidate for $x_{pulse}(t)$ is x(t) over any one period.

2. Define c(ω) to be the Fourier Integral transform of the pulse, scaled by $1/T_1$:

$$c(\omega) \equiv (1/T_1)X_{pulse}(\omega) = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-i\omega t} . \qquad (14.8) \text{ and } (1.1)$$

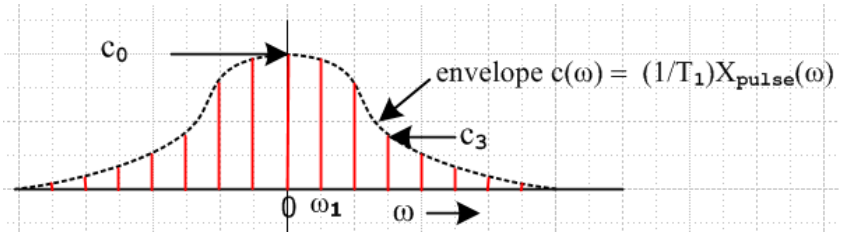
3. Then the Fourier Integral transform of the Pulse *Train* is as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\ \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1) \qquad (14.9)$$

where $c_m = c(\omega_m)$, $\omega_m = m\omega_1$, $\omega_1 = 2\pi/T_1$.

4. These c_m are the same c_m which appear in the next section. That is, they the complex Fourier Series coefficients.

Item 3 is our main result. It says that the Fourier Transform spectrum of an infinite sequence of pulses is a sum of equally-spaced delta function spikes whose coefficients are given by the continuous spectrum of the central pulse evaluated at the spike frequencies $\omega = m\omega_1$. The pulse spectrum $c(\omega) = (1/T_1)X_{pulse}(\omega)$ is normally thought of as the "coefficient envelope", while the equally spaced delta function spikes are the "lines". In this sample symbolic drawing of a spectrum, the infinitely-high delta function spikes of the spectrum are represented by finite vertical red line segments whose heights are the coefficients c_m. The red lines *are* the spectrum, and they track the envelope c(ω).



It may happen that certain c_m vanish, meaning that such lines are not present.

The item 3 sum includes the DC line m=0 having $\omega_0 = 0$. Unless c_0 happens to vanish, the pulse train has a DC component. As noted in Section 10 above, $X_{pulse}(0)$ is the area under $x_{pulse}(t)$. Only if this area is zero, do we get $c_0 = c(0) = (1/T_1)\, X_{pulse}(0) = 0$.

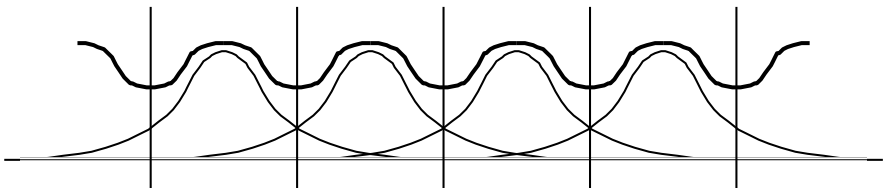
Whereas the Fourier Transform spectrum of a single pulse (localized, non-periodic) is continuous in ω, that of an infinite sequence of pulses is entirely discrete and has no continuous portions. This conforms with the well-known fact that the spectrum of any periodic function is discrete. In fact, we have just proven this to be so.

Note on $x_{pulse}(t)$. In our summary box above, we say that if x(t) is some *known* periodic function, one can take as a *candidate* for $x_{pulse}(t)$ the function x(t) restricted to any one period. In this case, the dt integration endpoints for the projection $X_{pulse}(\omega)$ only cover that selected period. If we select the period centered at t=0, then item 2 in the above summary box becomes perhaps more familiar:

$$c(\omega) = (1/T_1)X_{pulse}(\omega) = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-i\omega t} \qquad (14.13)$$

What is perhaps less obvious is that there are really many different candidates for $x_{pulse}(t)$ that result in the *same* x(t) pulse train. These other choices for $x_{pulse}(t)$ are pulses which slop over into more than one period T_1. When a pulse train is formed with such pulses, the pulses overlap.

To see how this might work, think of a pulse which has a nice gaussian shape and goes about half way into each neighboring T_1 interval. Draw some of these, then add them up to make the sum curve x(t). In this case, for your candidate $x_{pulse}(t)$, you can use either the gaussian, which overlaps into several intervals, or you can use one interval's worth of the sum curve x(t) (shown as the darker curve)



We have tried to keep our formulas completely general to allow for pulse trains formed from pulses which overlap into more than one period. This note applies to all the analysis and summary boxes which follow. The resulting spectrum is of course the same no matter which $x_{\mathbf{pulse}}(t)$ is chosen.

To summarize, we can write c(ω) in two equivalent ways

$$c(\omega) = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-i\omega t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{\mathbf{pulse}}(t)\, e^{-i\omega t} \qquad (14.14)$$

If we evaluate (14.14) at the discrete spike frequencies $\omega = m\omega_1$, we get

$$c_m = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-im\omega_1 t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{\mathbf{pulse}}(t)\, e^{-im\omega_1 t} \qquad (14.15)$$

Now since $e^{-im\omega_1(t+T_1)} = e^{-im\omega_1 t}\, e^{-im\omega_1 T_1} = e^{-im\omega_1 t}\, e^{-im2\pi} = e^{-im\omega_1 t}$, function $e^{-im\omega_1 t}$ is periodic with period T_1. Since x(t) in the first integral in (14.15) is also assumed periodic with period T_1, the integration can be over any interval of width T_1, so one usually takes this interval to be $(0,T_1)$. Thus,

$$c_m = (1/T_1) \int_{0}^{T_1} dt\, x(t)\, e^{-im\omega_1 t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{\mathbf{pulse}}(t)\, e^{-im\omega_1 t} \qquad (14.16)$$

Amplitude Modulated Pulse Train Notes

$$W(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{n=-N}^{N} y_n\, e^{+i\omega t_n}$$

Chapter 6: Statistical Pulse Trains

32. The Autocorrelation Function

This is a very simple idea which is often made to seem complicated. Starti with a reasonable function x(t). Define the autocorrelation function of x(t) as follows:

$$r_{\mathbf{x}}(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \qquad (32.1)$$

The integrand is the function evaluated at time t' times the same function evaluated at later time t'+t. Some texts might define the above with an extra overall "normalizing" constant factor; we leave it as shown.

A simple reflection property follows from the above definition (use t" = t' + t):

$$r_x(-t) = r_x(t) \qquad (32.2)$$

Thus, in (32.1) we could put either +t or -t in the last parentheses.

Although we have not yet mentioned statistics and randomness, one could easily imagine the following situation. Suppose x(t') is some sort of random function ("noise") that takes values in the range -1 to 1. It seems likely that for a value of the separation t that is larger than some small value, you might get $r_x(t) = 0$. The vague argument would be that there is no "correlation" between x(t') and x(t'+t), so the product of these two functions ought be be pretty random, and a sum of random numbers in the range -1 to 1 ought to be zero. Even in this case, we can see that the result is not zero if t = 0, since we are then summing a positive quantity. In fact, $r_x(0)$ is the area under $x(t)^2$.

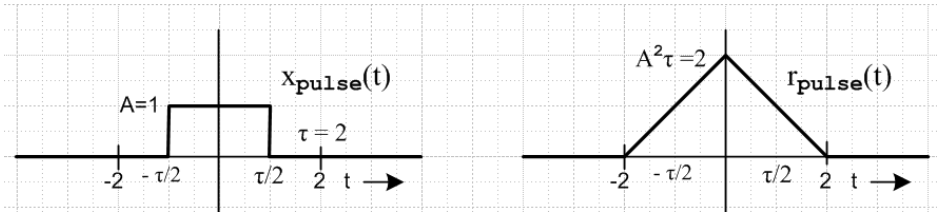
(a) Autocorrelation function for a Square Pulse.

Before going any further, let us compute the autocorrelation function for some simple case we are familiar with. A good candidate is x(t) = a square pulse of width τ and amplitude A. As in (9.1),

$$x_{pulse}(t) = A\,[\,\theta(t + \tau/2) - \theta(t - \tau/2)\,] \quad . \qquad (9.1)$$

One can easily do the above integral (32.1) to get the answer, but it is very obvious what the answer is. We are multiplying a box times a box shifted by t. Where they overlap, the integrand is A^2. The boxes only overlap if the absolute value of shift t is less than the width τ of the pulse. If this is so, the size of the overlap is τ-|t|. If |t| is larger than τ, there is no overlap, so the integral is 0. Thus,

$$r_{pulse}(t) \quad = A^2(\tau-|t|)\,\theta(\tau-|t|) \quad = (A^2\tau)\,[\,1 - |t|/\tau\,]\,\theta(\tau-|t|) \qquad (32.3)$$



(b) Energy in a finite signal x(t)

The total energy in a finite duration signal x(t) can be computed in either the t-domain or the ω-domain using using Parseval's formula (10.5), to which we add 1/R to each side.

$$\int_{-\infty}^{\infty} dt\,|x(t)|^2/R = \int_{-\infty}^{\infty} d\omega\,\frac{|X(\omega)|^2}{2\pi R} \qquad (32.4)$$

If we think of x(t) as the voltage across a resistor R, then dr $x^2(t)/R$ is the energy delivered to the resistor in time dt, and the integral on the left is the total energy in signal x(t). The dimensions on the right are, looking at (1.1),

$$d\omega \frac{|X(\omega)|^2}{2\pi R} = \text{sec}^{-1}\ (\text{volt-sec})^2/\text{ohms} = (\text{volt}^2/\text{ohms})\text{sec} = \text{watts-sec} = \text{joules} = \text{energy} \qquad (32.5)$$

Setting R = 1Ω, we can write this as

$$\int_{-\infty}^{\infty} dt\ p(t) = \int_{-\infty}^{\infty} d\omega\ \mathcal{P}(\omega) \qquad (32.6)$$

$p(t) \equiv |x(t)|^2$ = energy density in t-domain $p(t)dt$ = energy in dt

$\mathcal{P}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain $\mathcal{P}(\omega)d\omega$ = energy in dω

We shall refer to $\mathcal{P}(\omega)$ as the **spectral energy density** of signal x(t) whose Fourier Transform is X(ω). The isolated single pulse $x_{\mathbf{pulse}}(t)$ has a corresponding $\mathcal{P}_{\mathbf{pulse}}(\omega)$.

(b) The Spectral Density Connection: Wiener-Khintchine

Changing to t" = -t', we can trivially rewrite the definition (32.1) as follows:

$$r_{\mathbf{x}}(t) = \int_{-\infty}^{\infty} dt''x(t - t'')\ x(-t'') \ . \qquad \text{// energy units} \qquad (32.7)$$

This has the standard convolution equation form (3.1) where we select

$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$

$c(t) = x(-t) \quad \leftrightarrow \quad C(\omega) = X(-\omega) = X(\omega)^*$ // from (7.1) and (7.2)

Thus, the diagonalized frequency domain form (3.6) is

$$R_{\mathbf{x}}(\omega) = |X(\omega)|^2 \qquad (32.8)$$

Dividing by 2π we find that

$$\mathcal{P}(\omega) = (1/2\pi)\ R_{\mathbf{x}}(\omega) \qquad (32.9)$$

This is the key result. It says that the Fourier integral transform of the autocorrelation function of x(t) is 2π times the spectral energy density of x(t). This seemingly simple result gets the elaborate label of a Wiener-Khintchine Relation, see Bennett and Davey, *Data Transmission*, p 334.

Note also from (32.1) that $r_{\mathbf{x}}(t)$ evaluated at t = 0 gives the total energy in signal x(t),

$$r_x(0) = \int_{-\infty}^{\infty} dt\, x^2(t) \equiv E_x = \text{total energy in signal } x(t) \qquad (32.10)$$

For us, the significance of (32.9) is that we can "inject statistics" into a computation of the autocorrelation function, and then we will know the power spectrum of our statistical signal from (32.9). All we have to do is Fourier transform the autocorrelation function $r_x(t)$. Examples will follow.

(c) Verification of Wiener-Khintchine for a Square Pulse.

In (32.3) we have the autocorrelation function for a square pulse. One can insert this into the Fourier transform (1.1) to compute $R_x(\omega)$.

$$R_x(\omega) = \int_{-\tau}^{\tau} dt\, (A^2\tau)\, [\, 1 - |\, t\, |\, /\, \tau\,]\, e^{-i\omega t} = 2(A^2\tau) \int_0^{\tau} dt\, (1-t/\tau) \cos(\omega t)$$

$$= 2(A^2\tau)\, (1-\cos(\omega\tau))/(\omega^2\tau) = 4(A^2\tau)\, \sin^2(\omega\tau/2)/(\omega^2\tau) = (A^2\tau^2)\, \sin^2(\omega\tau/2)/(\omega\tau/2)^2$$

$$= (A\tau)^2\, [\, \mathrm{sinc}(\omega\tau/2)\,]^2 \qquad (32.10)$$

and this is recognized from (9.2) to be $|X(\omega)|^2$ for the square pulse, in agreement with (32.8) .

ok to here

33. Autocorrelation Function for a Pulse Train

(b) Spectral Energy Density of a Finite Pulse Train

Recall these equations from the end of Section 14,

$$x(t) = \sum_{n=-N}^{N} x_{pulse}(t - nT_1) \qquad (14.1)$$

$$X(\omega) = X_{pulse}(\omega)\, 2\pi\, \delta_5(\omega T_1, N) \qquad (14.4)$$

where δ_5 is a certain delta function model described in Appendix A. We then have

$$|\, X(\omega)\, |^2 = |\, X_{pulse}(\omega)\, |^2\ [2\pi\, \delta_5(\omega T_1, N)]^2 \qquad (33.11)$$

Recalling the definition of the δ_6 delta function model from Appendix A,

$$\delta_6(k,N) \equiv \frac{1}{2\pi}\ \frac{[2\pi\delta_5(\omega T_1, N)]^2}{(2N+1)} \qquad (A.20)$$

we obtain

$$\frac{1}{2\pi}\ \frac{|X(\omega)|^2}{(2N+1)} = |\, X_{pulse}(\omega)\, |^2\ \ \delta_6(\omega T_1, N) \qquad (33.12)$$

But the left side is $\mathcal{P}_1(\omega)$ as defined in (33.6) so we obtain

$$\mathcal{P}_1(\omega) = | X_{pulse}(\omega) |^2 \ \delta_6(\omega T_1, N) \tag{33.13}$$

for a finite pulse train. Recalling now that

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \tag{A.21}$$

we obtain for an infinite pulse train

$$\mathcal{P}_1(\omega) = | X_{pulse}(\omega) |^2 \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) = \sum_{m=-\infty}^{\infty} | X_{pulse}(m\omega_1) |^2 \delta(\omega T_1 - 2\pi m)$$

and this agrees with (33.7) derived in the previous section.

$$\mathcal{P}_1(\omega) = | X_{pulse}(\omega) |^2 \ \delta_6(\omega T_1, N)$$

so that

$$\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi} = | X_{pulse}(\omega) |^2 \ 2\pi [\ \delta_5(\omega T_1, N)]^2 \tag{33.3}$$

We now define

$$\mathcal{P}_1(\omega) \equiv \frac{1}{2N+1} \mathcal{P}(\omega) \tag{33.6}$$

which we can interpret as the spectral energy density of an average pulse in the pulse train, or as the average spectral energy density of the pulse train overall, it is the same either way. Then (33.3) reads,

$$\mathcal{P}_1(\omega) = \frac{1}{2N+1} | X_{pulse}(\omega) |^2 \ 2\pi [\ \delta_5(\omega T_1, N)]^2 \tag{33.7}$$

Example: We can estimate the sharpness of the delta function line away from the limit. Suppose we run a signal into our HP Spectrum Analyzer, and suppose it consumes 1 msec for its computation of each spectral "bin". If the signal under test is a 1 GHz pulse train, then $N = 10^6$ pulses of the pulse train are

analyzed for each bin. As shown in Appendix A, the peaks of (33.4) all have the same shape since the function is periodic, and the width of a peak is roughly $\Delta k = 2\pi/N$. Applying this to the δ in (33.7), the width of one of the δ lines is then $\Delta(\omega T_1) = 2\pi/N$ so that $\Delta\omega = \omega_1/N$. Our line width is thus

$$\Delta\omega = \omega_1/N \qquad \text{or} \qquad \Delta f = f_1/N \ . \tag{33.8}$$

In this case, the line width Δf would be $10^9/10^6 = 1$ kHz. This is just an application of the uncertainty principle which says $\Delta t\, \Delta f \approx 1$. Here, Δt is the time of measurement (the 1 msec), and Δf is the line width (the 1 KHz). The frequency is uncertain by amount Δf, so the line has width Δf.

(b) Autocorrelation function for a Pulse Train

We can compute the autocorrelation function for pulse train x(t) directly in the time domain using definition (32.1), and we shall do this later with the addition of statistics. Here instead we inverse Fourier transform both sides of (33.3),

$$\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi} = |\, X_{\mathbf{pulse}}(\omega)\,|^2 \ 2\pi\,[\,\delta_5(\omega T_1, N)]^2 \tag{33.3}$$

Using (1.2) apply $\int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}$ to both sides to get

$$\int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}\, \frac{|X(\omega)|^2}{2\pi} = \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega)|^2 \ 2\pi\,[\,\delta_5(\omega T_1, N)]^2$$

From (32.8) we know that $|X(\omega)|^2 = R_{\mathbf{x}}(\omega)$, so from (1.2) the left side is just $r_{\mathbf{x}}(t)$. On the right side the delta function pins the phasor ω to a specific value and the result is

$$\frac{r_{\mathbf{x}}(t)}{2\pi\delta(0)} = (1/T_1) \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2\, e^{im\omega_1 t} \tag{33.9}$$

Undoing the limit,

$$\frac{r_{\mathbf{x}}(t)}{(2N+1)} = (1/T_1) \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2\, e^{im\omega_1 t} \tag{33.10}$$

In this undone limit, we define the duration of the pulse train to be

$$T \equiv (2N+1)T_1 \tag{33.11}$$

Dividing both sides of (33.10) by T_1 then gives

$$\frac{r_x(t)}{T} = (1/T_1)^2 \sum_{m=-\infty}^{\infty} |X_{pulse}(m\omega_1)|^2 \, e^{im\omega_1 t} \qquad (33.12)$$

But we know that $(1/T_1)\, X_{pulse}(m\omega_1) = c(m\omega_1) = c_m$, the complex Fourier coefficient of (14.13). So we can rewrite the above result one more time to get a simple result for the autocorrelation function of x(t),

$$\frac{r_x(t)}{T} = \sum_{m=-\infty}^{\infty} |c_m|^2 \, e^{im\omega_1 t} \qquad (33.13)$$

Finally, we can set t=0 to get,

$$\frac{r_x(0)}{T} = \sum_{m=-\infty}^{\infty} |c_m|^2 \qquad \text{// average power} \qquad (33.14)$$

Recall from (32.10) that $r_x(0)$ is the total energy in the pulse train, $r_x(0)/T$ is the *average power* carried by the pulse train (delivered to a 1Ω load). Some texts add (1/T) in their definition of $r_x(t)$, but we have chosen not to do that.

So, the average power in a pulse train is the sum of the squared magnitudes of the complex Fourier Series coefficients. We know that $c_{-m} = c_m^*$ when x(t) is real, so we can reflect the negative sum over to the positive side and gain a factor of 2. We also know that $2c_m = a_m - ib_m$ from (15.4), so this gives us several other ways to write the above sum

$$\frac{r_x(0)}{T} = |c_0|^2 + 2 \sum_{m=1}^{\infty} |c_m|^2$$

$$= (1/4)\,|a_0|^2 + (1/2) \sum_{m=1}^{\infty} \{\, |a_m|^2 + |b_m|^2 \,\} \qquad (33.15)$$

The magnitude-squares of the Fourier Series coefficients give a power decomposition of a pulse train x(t).

34. Statistical Pulse Trains

Suppose we have, in place of the infinite pulse train of (33.1) with all coefficients unity, a new pulse train where the coefficients are completely general numbers, presumably real. Later we might want to have each coefficient be either a 1 or a 0. For now, we stay general, so here is our amplitude moduldate pulse train,

$$x(t) = \sum_{n=-N}^{N} a_n \; x_{pulse}(t - nT_1) \quad . \tag{34.1}$$

Now, lets imagine that we have a "statistical ensemble" of such pulse trains. Each such pulse train gets a superscript i as its label. Then we can say,

$$x^i(t) = \sum_{n=-N}^{N} a^i_n \; x_{pulse}(t - nT_1) \quad . \tag{34.2}$$

We now define the operation < > to indicate the average of some quantity over a large statistical ensemble of systems. Assume there are ' I ' systems in the ensemble. We then have,

$$\langle r_x(t) \rangle = (1/I) \sum_{i=1}^{I} \int_{-\infty}^{\infty} dt' \; x^i(t') \, x^i(t' + t) \quad . \tag{34.3}$$

If we now insert (34.2) twice into (33.3),

$$\langle r_x(t) \rangle = (1/I) \sum_{i=1}^{I} \int_{-\infty}^{\infty} dt' \; \{ \sum_{m=-N}^{N} a^i_m \; x_{pulse}(t' - mT_1) \} \{ \sum_{n=-N}^{N} a^i_n \; x_{pulse}(t' + t - nT_1) \}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} \{ (1/I) \sum_{i=1}^{I} a^i_m \, a^i_n \} \int_{-\infty}^{\infty} dt' \; x_{pulse}(t' - mT_1) \, x_{pulse}(t' + t - nT_1)$$

Following our statistical ensemble averaging rule, we define

$$\langle a_m a_n \rangle \equiv (1/I) \sum_{i=1}^{I} a^i_m \; a^i_n \tag{34.4}$$

so that

$$\langle r_x(t) \rangle = \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \int_{-\infty}^{\infty} dt' \; x_{pulse}(t' - mT_1) \, x_{pulse}(t' + t - nT_1) \tag{34.5}$$

Now replace integration variable t' by t" = t' - mT_1 to evalaute the integral appearing above,

$$\int_{-\infty}^{\infty} dt'' \, x_{\mathbf{pulse}}(t'') \, x_{\mathbf{pulse}}(t'' + \{ t + (m-n)T_1 \}) = r_{\mathbf{pulse}}[t + (m-n)T_1] \tag{34.6}$$

and we end up with

$$\langle r_{\mathbf{x}}(t) \rangle = \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \, r_{\mathbf{pulse}}[t + (m-n)T_1)] \tag{34.7}$$

This expresses the autocorrelation function of the pulse *train* as a sum, with statistically averaged coefficients, over the autocorrelation function of the pulse used to build the pulse train.

We wish to Fourier transform (34.7) to the ω domain using (1.1), so apply $\int_{-\infty}^{\infty} dt \, e^{-i\omega t}$ to both sides,

$$\int_{-\infty}^{\infty} dt \, e^{-i\omega t} \langle r_{\mathbf{x}}(t) \rangle = \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \int_{-\infty}^{\infty} dt \, e^{-i\omega t} \, r_{\mathbf{pulse}}[t + (m-n)T_1)]$$

On the left move the integration inside the statistical sum implied by <...> to get $\langle R_{\mathbf{x}}(\omega) \rangle$. On the right, replace integration variable t by $t' = t + (m-n)T_1$ so that $e^{-i\omega t} = e^{-i\omega t'} \, e^{i\omega (m-n) T_1}$. The result is

$$\langle R_{\mathbf{x}}(\omega) \rangle = R_{\mathbf{pulse}}(\omega) \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \, e^{i\omega (m-n) T_1} \tag{34.8}$$

We now use the relation (32.8) to replace each R(ω) with it's corresponding spectral density. Thus,

$$\langle |X(\omega)|^2 \rangle = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \, e^{i\omega (m-n) T_1} \tag{34.9}$$

Divide both sides by 2π and use (32.6) to write this as

$$\langle \mathscr{P}(\omega) \rangle = \mathscr{P}_{\mathbf{pulse}}(\omega) \sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle a_m a_n \rangle \, e^{i\omega (m-n) T_1} \tag{34.10}$$

This result describes the spectral energy density of a statistical pulse train in terms of the spectral energy density of the pulse we select to build the train, and in terms of the statistically averaged coefficients. The integral $\int_{-\infty}^{\infty} d\omega \, \mathscr{P}_{\mathbf{pulse}}(\omega)$ is the total energy in one of the pulses from which the pulse train is constructed, while $\int_{-\infty}^{\infty} d\omega \, \langle \mathscr{P}(\omega) \rangle$ is the total energy in the average pulse train of a statistical ensemble of pulse trains. If $N = \infty$, then this integral is infinite as well.

If we now make some statistical assumptions about the coefficients, we will arrive at the spectrum of a statistical pulse train. This is exactly what we want to know.

We already see from (34.9) a principle fact. The spectral density of the pulse acts as an "envelope" function, regardless of what happens with the double summation business. More on this below.
ok to here

35. Dealing with the statistical coefficient averages

By our definition (34.4) above, we have this average over our statistical ensemble of pulse *trains*:

$$\langle a_m a_n \rangle \equiv (1/I) \sum_{i=1}^{I} a^i_m a^i_n \quad . \tag{35.1}$$

A very simple case to consider is this:

$$\begin{aligned} a_m &= A \qquad \text{probability } p \\ a_m &= B \qquad \text{probability } 1\text{-}p \end{aligned} \tag{35.2}$$

Remember that index m labels a particular pulse in the pulse train. We can now state the averaged coefficients as follows, first for two different pulses, then for the same pulse:

$$\begin{aligned} n \neq m \qquad & \langle a_m a_n \rangle = [pp]\, AA + [p(1\text{-}p)]\, AB + [(1\text{-}p)p]BA + [(1\text{-}p)(1\text{-}p)]BB \equiv \alpha \\ n = m \qquad & \langle a_m^2 \rangle = [p]AA + [(1\text{-}p)]\, BB \equiv \beta \end{aligned} \tag{35.3}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. The reader must now stop reading and stare at the above equations until they make complete sense. Notice that the second line is quite distinct from the first line. In the double summation in (34.9), there are both "diagonal" terms where m = n, and off diagonal terms where m≠n. These groupings must be treated separately according to the above. We have defined new symbols α and β to emphasize that, for the situations shown, there is no longer any dependence on the n and m indices for these quantities.

Finite Pulse train with Random Coefficients

We repeat the previous section for a finite random pulse train. Using the above forms for the coefficient averages, (34.13) becomes

$$\langle |X(\omega)|^2 \rangle = |X_{pulse}(\omega)|^2 \quad \{ \; \alpha \sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + \beta \sum_{n=-N}^{N} \sum_{m=n} [1] \; \} \tag{35.9}$$

To evaluate the first double sum, we write it as

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = \sum_{n=-N}^{N} \{ \sum_{m=-N}^{N} [\, e^{i\omega(m-n)T_1}] - 1 \}$$

$$= \Big(\sum_{n=-N}^{N} e^{+i\omega n T_1} \Big) \Big(\sum_{m=-N}^{N} e^{-i\omega m T_1} \Big) - \sum_{n=-N}^{N} 1 = \Big| \sum_{n=-N}^{N} e^{+i\omega n T_1} \Big|^2 - (2N+1)$$

From Appendix A we have

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} = 2\pi\delta_5(k,N)$$

so the first double sum in (35.9) becomes

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = [2\pi\delta_5(\omega T_1,N)]^2 - (2N+1) \quad . \tag{35.10}$$

The second double sum in (35.9) is just

$$\sum_{n=-N}^{N} \sum_{m=n} [1] = \sum_{n=-N}^{N} [1] = (2N+1)$$

and therefore we may write (35.9) as

$$\langle |X(\omega)|^2 \rangle = |X_{pulse}(\omega)|^2 \{ \alpha \, [\, [2\pi\delta_5(\omega T_1,N)]^2 - (2N+1) \,] + \beta (2N+1) \}$$

$$= (2N+1) \; |X_{pulse}(\omega)|^2 \{ \alpha \, [\frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} - 1] + \beta (2N+1) \}$$

so

$$\frac{\langle |X(\omega)|^2 \rangle}{(2N+1)} = |X_{pulse}(\omega)|^2 \; \{(\beta-\alpha) + \alpha \; \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \} \tag{35.11}$$

The factor multiplying α can be replaced using (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \; \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} = \frac{1}{2N+1} \; \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \tag{A.20}$$

to give

$$\frac{\langle |X(\omega)|^2 \rangle}{(2N+1)} = |X_{pulse}(\omega)|^2 \; \{ (\beta-\alpha) + \alpha \, 2\pi \, \delta_6(\omega T_1,N) \} \tag{35.12}$$

From (33.1) and (33.2) we identify the left side with $2\pi \, \mathcal{P}_1(\omega)$. Dividing by 2π then gives

$$\langle \mathcal{P}_1(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \, [\, (\beta-\alpha) + \alpha \, 2\pi \, \delta_6(\omega T_1,N) \,] \tag{35.13}$$

which is the finite pulse train version of (35.8). In the limit $N \rightarrow \infty$ we know from (A.21)

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \tag{A.21}$$

so that (35.13) becomes

$$<\mathcal{P}_1(\omega)> = \mathcal{P}_{pulse}(\omega)\,[\,(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\,\delta(k-2\pi m)\,]$$

which reproduces (35.8).

$$\frac{<|X(\omega)|^2>}{(2N+1)} = |X_{pulse}(\omega)|^2 \{ (1/4) \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} + (1/4) \} \qquad (35.7)$$

Now we divide both sides by 2π. The left side then becomes $<\mathcal{P}(\omega)>/(2N+1) = <\mathcal{P}_1(\omega)>$ as in (33.6). And $|X_{pulse}(\omega)|^2/2\pi = \mathcal{P}_{pulse}(\omega)$ so,

$$<\mathcal{P}_1(\omega)> = \mathcal{P}_{pulse}(\omega) \{ \alpha \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} + (\beta-\alpha) \} \qquad (35.8)$$

Now take the large N limit and what happens?

What is the limit of $\frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)}$?? In Appendix A I consider large N only and look only at the peak at ω = 0 and integrate it from (-a,a), and that is how I show the area under δ_5 is 1 for that peak. Now consider this maple code

```
d5 := (1/(2*Pi))*sin((N+1/2)*k)/(k/2):
assume(a>0);
i := int(d5,k=-a..a);
```

$$i := 2\,\frac{\mathrm{Si}\left(\frac{1}{2}(2N+1)\,a\right)}{\pi}$$

```
limit(i,N=infinity);
```

$$1$$

```
i := int(d5^2,k=-a..a);
```

$$i := \frac{-1+\cos((2N+1)\,a)+\mathrm{Si}((2N+1)\,a)\,a+2\,\mathrm{Si}((2N+1)\,a)\,a\,N}{a\,\pi^2}$$

```
limit(i/(2*N+1),N=infinity);
```

$$\frac{1}{2}\frac{1}{\pi}$$

This tells me that

$$\lim_{N\to\infty} \int_{-a}^{a} \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} = (2\pi)^2\,(1/2\pi) = 2\pi$$

Therefore I can regard

$$\lim_{N\to\infty} \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} = 2\pi\delta(\omega T_1) \text{ for the central peak only, and more generally}$$

$$= 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega T_1 - 2\pi n)$$

Then (35.8) becomes

$$\langle \mathcal{P}_1(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \{ \alpha \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} + (\beta-\alpha) \} \quad (35.8)$$

$$= \mathcal{P}_{pulse}(\omega) \{ \alpha\, 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega T_1 - 2\pi n) + (\beta-\alpha) \}$$

And this agrees with my all delta function approach. But now I have a result for finite N as well.

Here $\mathcal{P}_{pulse}(\omega)$ is the spectral energy denstiy of $x_{pulse}(t)$, and its integral over dω is finite. The left side is $\langle \mathcal{P}_1(\omega) \rangle$ which is the statistically averaged energy density per pulse for the pulse train, and its integral over dω is also finite. Note that no N's or δ(0)'s appear explicitly in this result! If N = ∞, $\delta(\omega T_1 - 2\pi m)$ is a true delta function, but if N is large but finite, then $\delta(\omega T_1 - 2\pi m)$ is given by (33.4).

Spectral Energy Density of a Random Pulse Train (35.9)

$$\langle \mathcal{P}_1(\omega) \rangle = \mathcal{P}_{pulse}(\omega) [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)] \quad \text{// per pulse} \quad (35.8)$$

$$\alpha = [pp]\, AA + [p(1-p)]\, AB + [(1-p)p]BA + [(1-p)(1-p)]BB \quad \text{// } = \langle a_m a_n \rangle \text{ when } n \neq m$$

$$\beta = [p]AA + [(1-p)]\, BB \quad \text{// } = \langle a_m a_n \rangle \text{ when } n = m \quad (35.3)$$

A pulse has probably p of having amplitude A, and probability 1-p of having amplitude B.

T_1 = pulse spacing.

For coefficients A=1, B=0, $\alpha = p^2$, $\beta = p$, $(\beta - \alpha) = p(1-p)$, where p = probability of a coded 1.

This is a fascinating result for $\langle \mathcal{P}_1(\omega) \rangle$. There are two weighted terms: the first term has weight $(\beta - \alpha)$, the second term has weight α. The first term is just $(\beta-\alpha)$ times the spectrum of the pulse used to build the pulse train -- it is a continuous function of ω. The second term is the same discrete spectrum we found in (33.3) for the regular pulse train. It's delta functions pick off specific values of $|X_{\mathbf{pulse}}(\omega)|^2$ at the lines.

Looking back at (35.3), we see that if the pulse train coefficients are A=1 or B=0, then $\alpha = p^2$ and $\beta = p$, so that $(\beta - \alpha) = p(1-p)$. Here are a few examples using A = 1 and B = 0.

Example 1: Let p=1 with A=1 and B=0. Then $\alpha = \beta = 1$. This is the non-statistical pulse train (33.1) where each pulse has a unity coefficient. Now having the experience with handling the delta functions, we can just stare at (33.2) and mentally square both sides and divide by 2π. The resulting $2\pi\delta(0)$ gets moved to the left side denominator, it is interpreted as (2N+1), so the left side becomes $\mathcal{P}_1(\omega)$, so (33.2) squared reads

$$\mathcal{P}_1(\omega) = \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1)\ 2\pi\ \delta(\omega T_1 - 2\pi m)$$

But this is exactly what (35.8) says when $\alpha = \beta = 1$. Since all pulses are the same, $\langle \mathcal{P}_1(\omega) \rangle = \mathcal{P}_1(\omega)$.

Example 2: Let p=0 with A=1 and B=0. Then every pulse has B = 0, $x(t) \equiv 0$, $\alpha = \beta = 0$, and both terms in (38.5) vanish so there is zero spectral energy density.

Example 3: Let p=1/2 with A=1 and B=0, so $\alpha = p^2 = 1/4$ and $\beta = p = 1/2$, This pulse train then has an equal probably of 1's and 0's. We find from (38.5) that

$$\langle \mathcal{P}_1(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[1/4 + 1/4 \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m) \right] \qquad (35.10)$$

This shows that the discrete spectrum has been reduced to 1/4 of its full strength, and a continuous spectrum exists with coefficient 1/4 as shown.

It would be interesting to try and confirm this last result using a spectrum analyzer for various pulses.

36. A View from the High Ground

In this Chapter we introduced some new definitions, such as the autocorrelation function, and the Wiener-Khintchine relation. We used these in our derivation of the statistical average of the spectral density of a pulse train.

The whole thing was somewhat of a deception, and here we want to make sure the reader has not missed this point. Our main purpose for introducing autocorrelation and Wiener-Khintchine was to get these well-known terms into our dialog, since they appear in texts.

In reality, there is nothing at all new in this Chapter, and we certainly did not need to use the words autocorrelation or Wiener-Khintchine to get our main results. As shown in (32.7), autocorrelation is just a standard convolution of x(t) with x(-t), and Wiener-Khintchine is just the convolution theorem (3.6) applied to this convolution equation. In effect, we could have just defined R(ω) to be $|X(\omega)|^2$, given it the new name "autocorrelation", and then forgotten about it.

Here for example is a simple derivation of (34.10):

Start off with the pulse train "i" with statistical coefficients a^i_n, as in (34.2),

$$x^i(t) = \sum_{n=-\infty}^{\infty} a^i_n \, x_{pulse}(t - nT_1) \tag{36.1}$$

Fourier transform both sides in the usual manner, picking up the usual time-shift phase,

$$X^i(\omega) = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} a^i_n \, e^{-in\omega T_1} \tag{36.2}$$

Magnitude-square this thing to get

$$|X^i(\omega)|^2 = |X_{pulse}(\omega)|^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a^i_m a^i_n \, e^{i\omega(m-n)T_1} \tag{36.3}$$

Finally, divide by 2π, average over ensemble index i, identify $\langle |X^i(\omega)|^2/2\pi \rangle = \langle \mathscr{P}(\omega) \rangle$ and we get

$$\langle \mathscr{P}(\omega) \rangle = \mathscr{P}_{pulse}(\omega) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n \rangle \, e^{i\omega(m-n)T_1}$$

which is (34.10).

The dervivation of (35.8) would then proceded as outlined in Section 35.

A paradox and its resolution

Now while we are here, we can back up and apply our statistical average directly to the spectrum in (36.2). Our result is

$$\langle X(\omega)\rangle = X_{\mathbf{pulse}}(\omega) \sum_{n=-\infty}^{\infty} \langle a_n\rangle \; e^{-in\omega T_1} \tag{36.4}$$

We could then argue that

$$\langle a_n\rangle = [p]\, A + [1-p]\, B \qquad = p \qquad \text{if } A=1,\ B=0 \qquad . \tag{36.5}$$

In thise case, we can extract p from the sum, which then collapses to form the usual (13.2) delta functions. The result is then the same as with the regular pulse train (33.2) with an overall factor of p out front, namely,

$$\langle X(\omega)\rangle \quad = \; p \sum_{m=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\; 2\pi\; \delta(\,\omega T_1 - 2\pi m) \tag{36.6}$$

Thus says that our average *spectrum* is 100% discrete, there is no continuous part! If p = 1/2, the average spectrum is just 1/2 times our unit amplitude pulse train spectrum (14.4). How can this be true, if we just showed in (35.10) that the average *spectral density* $\langle |X(\omega)|^2\rangle$ has a continuous spectral component?

The answer lies in the fact that $\langle ab\rangle \neq \langle a\rangle\langle b\rangle$, where $\langle\ \rangle$ is our averaging operation. Thus

$$\langle |X(\omega)|^2\rangle \neq \langle X(\omega)\rangle\langle X(\omega)^*\rangle \tag{36.7}$$

There is no reason in the world why the average of a product should be the product of the averages. The HP Spectrum Analyzer measures $\langle |X(\omega)|^2\rangle$, not $\langle X(\omega)\rangle$.

Idea: have Maple generate a long pulse train with p= 1/2, then have it compute the Fourier spectrum X(ω) and see if its spectrum looks discrete. Then maybe try to get it do the energy density and see if it seems to have a discrete and continuous piece. This would be quite interesting.

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