34. Statistical Amplitude Modulated Pulse Trains Part I

(a) Energy and Power Spectra for a General Pulse Train

In the previous Section we dealt with simple pulse trains. Here we consider the more general amplitude modulated pulse train. In all equations, one can replace $\sum_{n=-\infty}^{\infty}$ by $\sum_{n=-N}^{N}$ to adapt the equation to a finite pulse train instead of an infinite one.

$$x(t) = \sum_{n=-\infty}^{\infty} y_n \, x_{\mathbf{pulse}}(t - t_n) \tag{34.1}$$

$$X(\omega) = (1/T_1) X_{\mathbf{pulse}}(\omega)\, X'(\omega) = X_{\mathbf{pulse}}(\omega)\, X''(z) \tag{34.2}$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega n T_1}. \tag{34.3}$$

To find the frequency-domain power spectrum of a signal x(t), our first task is to compute $|X(\omega)|^2$. From (34.2) we get

$$|X(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 \, (1/T_1)^2 \, |X'(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 \, |X''(z)|^2 \qquad z = e^{i\omega T_1} \tag{34.4}$$

We therefore must deal with the following object, using (33.4),

$$|X''(z)|^2 = (1/T_1)^2 \, |X'(\omega)|^2 = \left| \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega n T_1} \right|^2$$

$$= \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega n T_1} \sum_{m=-\infty}^{\infty} y_m^* \, e^{+i\omega m T_1} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega (m-n) T_1} \tag{34.5}$$

so that

$$|X(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega (m-n) T_1} \tag{34.6}$$

From box (33.36) we then have

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = (1/2\pi) \, |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega (m-n) T_1} \tag{34.7}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = (1/2\pi T) \, |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega (m-n) T_1} \tag{34.8}$$

which we can write as

$$\mathcal{E}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \quad (34.9)$$

$$\mathcal{P}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \quad (34.10)$$

$$P = \int_{-\infty}^{\infty} d\omega \, \mathcal{P}(\omega) \quad . \qquad (33.31) \quad (34.11)$$

Not knowing details of the y_n there is not much else we can do in these expressions.

(b) Energy and Power Spectra for a Statistical Pulse Train

We imagine now a large ensemble of I pulse trains in which a particular pulse train is labeled by index i. This pulse train has coefficients $y_n^{(i)}$. We are interested in the ensemble averages of $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$ and P. We indicate the ensemble average of some quantity Q as <Q>. By averaging the last three equations above we obtain

$$\langle \mathcal{E}(\omega) \rangle = T_1 \mathcal{P}_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* y_n \rangle e^{i\omega(m-n)T_1} \quad (34.12)$$

$$\langle \mathcal{P}(\omega) \rangle = T_1 \mathcal{P}_{pulse}(\omega) (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* y_n \rangle e^{i\omega(m-n)T_1} \quad (34.13)$$

$$\langle P \rangle = \int_{-\infty}^{\infty} d\omega \, \langle \mathcal{P}(\omega) \rangle \quad . \quad (34.14)$$

where

$$\langle y_m^* y_n \rangle = (1/I) \langle y_m^{(i)*} y_n^{(i)} \rangle \quad (34.15)$$

Up to this point we have tried to be general, allowing the $y_m^{(i)}$ to be complex coefficients, but from now on we consider them to be real, so we can delete the asterisks in the above equations.

A very simple case to consider is this, where A and B are real,

$y_m = A$ probability p
$y_m = B$ probability 1-p (34.16)

Remember that index m labels pulse location m in the pulse train. If probabilities of a pulse value at locations m and at n are uncorrelated (they are independent of each other), then we can write the averaged coefficients as follows, first for two different locations $m \neq n$, then for the same location $m = n$:

$$n \neq m \qquad \alpha \equiv \langle y_m y_n \rangle = [pp]\ AA + [p(1-p)]\ AB + [(1-p)p]BA + [(1-p)(1-p)]BB$$

$$n = m \qquad \beta \equiv \langle y_m^2 \rangle = [p]AA + [(1-p)]\ BB \tag{34.17}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. The reader must now stop reading and stare at the above equations until they make complete sense. Notice that the second line is quite distinct from the first line. In the double summation in (34.13), there are both "diagonal" terms where m = n, and off diagonal terms where m≠n. These groupings must be treated separately according to the above. We have defined new symbols α and β to emphasize that, for the situations shown, there is no longer any dependence on the n and m indices for these quantities. In a later section, we shall encounter a situation where the above equations do not apply because there *is* correlation between positions m and n (the AMI line code).

Inserting (34.17) into (34.13) gives

$$\langle \mathcal{P}(\omega) \rangle = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T)\, \{\ \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \ + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \ \} \tag{34.18}$$

In Section 35 we shall evaluate this important result for infinite and finite pulse trains, but we pause momentarily for a digression.

(c) Digression on expectation values, correlation and covariance

In the language of statistics, one can imagine a "random variable" Y_m associated with location m in our pulse train, and this variable can take certain values y_m (perhaps 0 and 1). We imagine a statistical ensemble of pulse trains indexed by i, and at location m the i_{th} pulse train has value $y_m^{(i)}$ of Y_m.

If all pulse trains in our ensemble of pulse trains count the same in our average (they do), then one can write $\langle y_m y_n \rangle$ in (35.1) as

$$E(Y_m Y_n) = (1/I) \sum_{i=1}^{I} y_m^{(i)} y_n^{(i)} \quad \text{in the sense of} \quad E(XY) = (1/I) \sum_{i=1}^{I} x^{(i)} y^{(i)}$$

where E(q) means the expectation value of quantity q. This particular expectation value has a special name: it is called the **correlation** of X and Y. So in fact we have

$$\langle y_m y_n \rangle = E(Y_m Y_n) = (1/I) \sum_{i=1}^{I} y_m^{(i)} y_n^{(i)} \equiv \text{corr}(Y_m Y_n) \qquad \text{// correlation of } Y_m \text{ and } Y_n$$

If $\text{corr}(Y_m Y_n)$ is a significant positive or negative number, then the random variables Y_m and Y_n are strongly correlated in some way. This means that the probabilities of values of $y_m^{(i)}$ and $y_n^{(i)}$ are related in some way. One might wonder how there could be a correlation between locations m and n in a pulse train. We shall see this happen below in the discussion of the AMI lines code.

If Y_m and Y_n are independent random variables, we expect corr($Y_m Y_n$) ≈ 0.

Two other expectation values of interest are these, the **mean** and the **covariance** :

$$E(Y_m) = (1/I) \sum_{i=1}^{I} y_m^{(i)} = \langle y_m \rangle \qquad \text{// mean of } Y_m$$

$$E([Y_m - \langle y_m \rangle][[Y_n - \langle y_n \rangle]) = (1/I) \sum_{i=1}^{I} [y_m^{(i)} - \langle y_m \rangle] [y_n^{(i)} - \langle y_n \rangle]$$

$$= \text{cov}(Y_m Y_n) \qquad \text{// covariance of } Y_m \text{ and } Y_n$$

If it happens that $\langle y_m \rangle = 0$ for a certain pulse train, then covariance and correlation are the same.