New version of Section 33 (a) PhL 3.26.05

33. Energy and Power for a Pulse Train; using the Autocorrelation Function

In subsections (a) through (d) we deal only with simple pulse trains. Along the way, certain facts are developed which apply to general as well as simple pulse trains. In subsection (e) we gather together these general facts, and then show how quantities of interest can be related to the autocorrelation function.

(a) Infinite Simple Pulse Train

This is the first section in which we use the $\delta(0)$ notation of Appendix A which may make the reader feel a bit uncomfortable. In subsection (b) we shall repeat everything for a finite pulse train and then take the limit $N\to\infty$ to obtain the same results without using $\delta(0)$. In both sections we shall include the Z transform in passing, but our main work is in the ω variable, not the z variable.

From Section 14 (a) we know the spectrum of an infinite pulse train formed from pulses $x_{\mathbf{pulse}}(t)$ separated by time T_1 ,

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\mathbf{pulse}}(t - nT_1) \qquad (14.1)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m) \qquad (14.4)$$

We can obtain the same expressions from box (25.4) which summarizes amplitude modulated pulse trains by setting all amplitudes to $y_n = 1$ and changing w,W to x,X:

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\ x_{\mathbf{pulse}}(t - t_n) = \sum_{n=-\infty}^{\infty} x_{\mathbf{pulse}}(t - t_n) \qquad (33.1)$$

$$X(\omega) = (1/T_1)X_{\mathbf{pulse}}(\omega)\ X'(\omega) = X_{\mathbf{pulse}}(\omega)\ X''(z) \qquad (33.2)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} \quad . \qquad (33.3)$$

$X'(\omega)$ is the Digital Fourier Transform of x(t), and X"(z) is the Z Transform, where $z = e^{i\omega T_1}$

In the last line we then use (13.2)

$$\sum_{n=-\infty}^{\infty} e^{\pm ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

so that (33.3) becomes

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.4}$$

and then (33.2) says

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\, X'(\omega) = X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.5}$$

in agreement with (14.4) quoted just above (33.1).

To find the frequency domain power spectrum of a signal x(t), our first task is to compute $| X(\omega) |^2$. From (33.2) we get

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2\, (1/T_1)^2\, |X'(\omega)|^2 = |X_{pulse}(\omega)|^2\, | X''(z) |^2 \qquad z = e^{i\omega T_1} \tag{33.6}$$

We therefore must deal with the following object, using (33.4),

$$| X''(z) |^2 = (1/T_1)^2\, |X'(\omega)|^2 = \left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right]^2 , \tag{33.7}$$

and we are now faced with the issue of squaring delta functions. Formally these objects don't exist in the realm of distribution theory, but as discussed in Appendix A, we can deal with them in an *ad hoc* way which proves to be useful.

$$\left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right]^2 = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(\omega T_1 - 2\pi n)$$

Looking at the product of the two delta functions, there can be no contribution to the double sum unless m = n, so we continue

$$= \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(0) = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

so that

$$\left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right]^2 = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \tag{33.8}$$

The object $\delta(0)$ is formally undefined, but in Appendix A we ascribe the meaning that $2\pi\delta(0) = 2N+1$ in the limit that $N \to \infty$ and we can always "undo the limit" when necessary. We shall firm up this idea in section (b) directly below. So we have shown then that

$$| X''(z) |^2 = (1/T_1)^2 |X'(\omega)|^2 = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.9}$$

or

$$\frac{|X''(z)|^2}{[2\pi\delta(0)]} = (1/T_1)^2 \frac{|X'(\omega)|^2}{[2\pi\delta(0)]} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.10}$$

Then from (33.6)

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 \frac{|X''(z)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 (1/T_1)^2 \frac{|X'(\omega)|^2}{[2\pi\delta(0)]}$$

$$= |X_{pulse}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.11}$$

We shall now repeat the above set of steps for a finite pulse train.

(b) Finite Simple Pulse Train

Our finite pulse train always has pulses ranging from n = -N to N instead of from n = -∞ to ∞. We start off exactly as in the previous section but with limited sums

$$x(t) = \sum_{n=-N}^{N} y_n \, x_{pulse}(t - t_n) = \sum_{n=-N}^{N} x_{pulse}(t - t_n) \tag{33.12}$$

$$X(\omega) = (1/T_1) X_{pulse}(\omega) \, X'(\omega) = X_{pulse}(\omega) \, X''(z) \tag{33.13}$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n \, e^{-i\omega n T_1} = \sum_{n=-N}^{N} e^{-i\omega n T_1} \tag{33.14}$$

In the last line we then use (13.3)

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi \, \delta_5(k,N) \qquad -\infty < k < \infty \qquad \textit{(13.3)}$$

where δ_5 is a periodic delta function model discussed in Appendix A (b). Equation (33.14) becomes

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi \, \delta_5(\omega T_1, N) \tag{33.15}$$

and then equation (33.13) says

$$X(\omega) = X_{pulse}(\omega) \, 2\pi \, \delta_5(\omega T_1, N) \, . \tag{33.16}$$

This δ_5 is periodic with period 2π and has identical peaks separated by 2π. For finite N, these are peaks of finite width and height. Squaring, we find

$$| X(\omega) |^2 = | X_{\mathbf{pulse}}(\omega) |^2 \; [2\pi \, \delta_5(\omega T_1,N)]^2 \quad . \tag{33.17}$$

Recalling the definition of the δ_6 delta function model from Appendix A,

$$2\pi \, \delta_6(k,N) \equiv \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} \tag{A.20}$$

we obtain

$$\frac{|X(\omega)|^2}{(2N+1)} = | X_{\mathbf{pulse}}(\omega) |^2 \; 2\pi \, \delta_6(\omega T_1,N) \tag{33.18}$$

and this is the finite pulse train result. We can then take the limit $N\to\infty$ and make use of

$$\lim_{N\to\infty} \delta_6(\omega T_1,N) = \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \tag{A.21}$$

to find that

$$\lim_{N\to\infty} \left[\frac{|X(\omega)|^2}{(2N+1)}\right] = | X_{\mathbf{pulse}}(\omega) |^2 \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \tag{33.19}$$

and this replicates (33.11) with the promised connection $2\pi\delta(0) = \lim_{N\to\infty} (2N+1)$.

It is useful now to provide some side-by-side comparisons of results

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.4)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi \, \delta_5(\omega T_1,N) \qquad \text{finite} \qquad (33.15)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.5)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \; 2\pi \, \delta_5(\omega T_1,N) \; . \qquad \text{finite} \qquad (33.16)$$

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.11)$$

$$\frac{|X(\omega)|^2}{(2N+1)} = |X_{\mathtt{pulse}}(\omega)|^2 \; 2\pi\,\delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N) \qquad \text{finite} \qquad (33.18)$$

One can interpret $\frac{|X(\omega)|^2}{[2\pi\delta(0)]}$ as the value of $|X(\omega)|^2$ *per pulse* in an infinite pulse train.

(c) Spectral Energy Density and Spectral Power Density in a Simple Pulse Train

In this section, everything is in the frequency domain, nothing is in the time domain.

Recall from (32.6) that

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$\mathcal{E}(\omega)d\omega$ = energy in dω (joules)

where $\mathcal{E}(\omega)$ is the spectral energy density of signal x(t) whose Fourier Transform is X(ω).

If we divide $\mathcal{E}(\omega)$ by 2N+1 or 2πδ(0) we obtain the pulse train's average spectral energy density per pulse, which is the same as the energy density of an average pulse in the pulse train. Using our two expressions above for infinite and finite pulse trains, we then find

$$\frac{\mathcal{E}(\omega)}{[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_{\mathtt{1}} - 2\pi m)$$

$$\frac{\mathcal{E}(\omega)}{(2N+1)} = \frac{|X(\omega)|^2}{2\pi(2N+1)} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \; 2\pi\,\delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N) \qquad (33.20)$$

If we divide the energy of the average pulse by $T_{\mathtt{1}}$, we obtain power of an average pulse, and this is the same as the average power of the pulse train. Thus,

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_{\mathtt{1}}[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi T_{\mathtt{1}}[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_{\mathtt{1}}) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_{\mathtt{1}} - 2\pi m)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_{\mathtt{1}}[2N+1]} = \frac{|X(\omega)|^2}{2\pi T_{\mathtt{1}}[2N+1]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_{\mathtt{1}})\;2\pi\,\delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N) \qquad (33.21)$$

If is perhaps helpful to define

$$T \equiv \begin{cases} [2\pi\delta(0)]T_{\mathtt{1}} & \text{infinite pulse train} \\ (2N+1)T_{\mathtt{1}} & \text{finite pulse train} \end{cases} \qquad (33.22)$$

Then the above maybe be restated

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_{\mathtt{1}}) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_{\mathtt{1}} - 2\pi m)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{\mathbf{pulse}}(\omega)|^2\ (1/T_1)\ 2\pi\ \delta_6(\omega T_1, N) \tag{33.23}$$

Meanwhile, our $x_{\mathbf{pulse}}(t)$ which lasts only for duration T_1 itself has an energy and a power

$$\mathcal{P}_{\mathbf{pulse}}(\omega) \equiv \frac{\mathcal{E}_{\mathbf{pulse}}(\omega)}{T_1} = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \tag{33.24}$$

We can then write (***) in the following compact form

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules}$$

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{\mathbf{pulse}}(\omega)\ 2\pi\ \delta_6(\omega T_1, N) \qquad \text{joules} \tag{33.25}$$

Notice that $\delta(\omega T_1 - 2\pi m)$ and $\delta_6(\omega T_1, N)$ are both dimensionless, so the dimensions in each equation trivially match. A power density $\mathcal{P}(\omega)$ has dimensions of energy = joules, so that $\mathcal{P}(\omega)d\omega$ then has the dimensions of joules/sec = watts, and this is the pulse train power contained in interval $d\omega$ of the spectrum.

For the infinite pulse train, we are always allowed to write

$$2\pi\ \delta(\omega T_1 - 2\pi m) = (2\pi/T_1)\ \delta(\omega - m(2\pi/T_1)) = \omega_1\ \delta(\omega - m\omega_1) \qquad \omega_1 \equiv 2\pi/T_1$$

to get

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \qquad \text{infinite} \tag{33.26}$$

which shows more explicitly that the power lines occur at the harmonics $\omega = m\omega_1$. Recall now these two earlier facts

$$c(\omega) \equiv (1/T_1)X_{\mathbf{pulse}}(\omega) \tag{14.14}$$

$$c_{\mathbf{m}} \equiv c(m\omega_1) = (1/T_1)X_{\mathbf{pulse}}(m\omega_1) \tag{14.10}$$

For the infinite pulse train we can then write

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1}\ \frac{2\pi}{T_1} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)$$

$$= \frac{|X_{\mathbf{pulse}}(\omega)|^2}{T_1^2} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = |c(\omega)|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} |c(m\omega_1)|^2\ \delta(\omega - m\omega_1)$$

so that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2 \ \delta(\omega - m\omega_1). \qquad (33.27)$$

Recall from box (15.12) that Dim(c_m) = Dim[x(t)] = volts (say), so $|c_m|^2$ = watts into a 1Ω resistor, and since $\delta(\omega - m\omega_1)$ has dimensions sec, $|c_m|^2 \ \delta(\omega - m\omega_1)$ then has dimensions watt-sec = joules, as befits any $\mathcal{P}(\omega)$ object.

If we assume x(t) is a real pulse train, then X(ω) is Hermitian X(-ω) = [X(ω)]* by (7.1), which means

$$c_{-m} = (1/T_1)X_{pulse}(-m\omega) = (1/T_1)[X_{pulse}(m\omega)]^* = c_m^*$$

so

$$|c_{-m}|^2 = |c_m|^2 \qquad \text{x(t) real} \qquad (33.28)$$

and then we can fold the negative part of the sum in (**) to get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2 \ \delta(\omega - m\omega_1) = |c_0|^2 \ \delta(\omega) + 2 \sum_{m=1}^{\infty} |c_m|^2 \ \delta(\omega - m\omega_1) \qquad (33.29)$$

and then finally, recalling from our Fourier Series box (15.2) that $c_m = [\ a_m - ib_m\ ]/2$ and $b_0 = 0$ we get

$$\mathcal{P}(\omega) = a_0^2 \ \delta(\omega) + (1/2) \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \ \delta(\omega - m\omega_1) \qquad (33.30)$$

(d) Average Power P for a Simple Pulse Train

We seek an expression for the average power P in a general pulse train. This is of course a time domain quantity, not a frequency domain quantity.

P = [total energy in pulse train / time duration of pulse train] = average pulse train power

$$= (1/T) \int_{-\infty}^{\infty} dt\ |x(t)|^2 = \int_{-\infty}^{\infty} d\omega\ \frac{|X(\omega)|^2}{2\pi T} \qquad \text{// from (32.4) with R = 1Ω}$$

so

$$P = \int_{-\infty}^{\infty} d\omega\ \mathcal{P}(\omega) \ . \qquad \text{// from (33.23)} \qquad (33.31)$$

This is certainly reasonable since $\mathcal{P}(\omega)$ is the average spectral power density of the pulse train.

For the special case of a simple pulse train, we found above that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2 \ \delta(\omega - m\omega_1) = a_0^2 \ \delta(\omega) + (1/2) \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \ \delta(\omega - m\omega_1)$$

Therefore the power in a simple pulse train is given by

$$P = \sum_{m=-\infty}^{\infty} |c_m|^2 = a_0^2 + (1/2) \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \qquad (33.32)$$

(e) General Pulse Train results and connection with the Autocorrelation Function

Certain results of the previous section apply to general pulse trains and we gather them here:

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = \text{energy density in the } \omega\text{-domain} \qquad \text{(joule-sec)} \qquad \textit{(32.6)}$$

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad \textit{(33.22)}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad \textit{(33.23)}$$

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \qquad \textit{(33.24)}$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) \quad . \qquad \textit{(33.31)}$$

We now bring the autocorrelation function into the discussion. Let x(t) be an arbitrary but real pulse train, and recall that

$$r_x(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \qquad \textit{(32.1)}$$

Then, using T from (33.22),

$$r_x(0) \equiv \int_{-\infty}^{\infty} dt'\, x(t')^2 = P\,T = E = \text{total energy in the pulse train} \qquad (33.33)$$

so the average pulse train power maybe written in terms of the autocorrelation function evaluated at t = 0,

$$P = r_x(0)/T \quad . \qquad (33.34)$$

We diagonalized (32.1) treated as a convolution equation to obtain

$$|X(\omega)|^2 = R_x(\omega)$$

which we called the Wiener-Khintchine Relation. Here $R_x(\omega)$ is the Fourier Integral Transform of the autocorrelation function $r_x(t)$ of x(t). Then from (33.23) that $\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T}$ we get

$$\mathcal{P}(\omega) = R_{\mathbf{x}}(\omega)/ (2\pi T) \qquad (33.35)$$

In this way, both P and $\mathcal{P}(\omega)$ can be expressed in terms of the autocorrelation function. Thus, one approach to finding P and $\mathcal{P}(\omega)$ for a pulse train is to try and determine $r_{\mathbf{x}}(t)$.

Here then is a box summarizing all the general pulse train results:

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$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad \textit{(33.22)}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = R_{\mathbf{x}}(\omega)/ (2\pi T) \quad \text{joules} \qquad \textit{(33.23) and (33.35)}$$

$$\mathcal{P}_{\mathbf{pulse}}(\omega) \equiv \frac{\mathcal{E}_{\mathbf{pulse}}(\omega)}{T_1} = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \qquad \textit{(33.24)}$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) = r_{\mathbf{x}}(0)/T \quad \text{watts} \qquad \textit{(33.31) and (33.34)}$$

If $P(f)df = \mathcal{P}(\omega)d\omega = \mathcal{P}(\omega)\, 2\pi df$, then $P(f) = 2\pi\mathcal{P}(\omega)$.