old sections 33,34,35 PhL 3.26.13

33. Autocorrelation and Spectral Energy Density for a Simple Pulse Train

(a) Infinite Simple Pulse Train

This is the first section in which we use the δ(0) notation of Appendix A which may make the reader feel a bit uncomfortable. In subsection (b) we shall repeat everything for a finite pulse train and then take the limit N→∞ to obtain the same results without using δ(0).

From summary box (14.12) we know the spectrum of an infinite pulse train formed from pulses $x_{\mathbf{pulse}}(t)$ separated by time T_1 [in the second line, note that $\delta(\omega T_1 - 2\pi m) = (1/T_1)\,\delta(\omega - m\omega_1)$]

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\mathbf{pulse}}(t - nT_1) \qquad (14.1)$$

$$X(\omega) = \sum_{m=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\; 2\pi\; \delta(\omega T_1 - 2\pi m) \qquad (14.4)$$

We have been careful to write this in its original developed form without any scaling of the delta function, heeding the warning below equation (A.30) in Appendix A. The spectrum of the pulse is being sensed at discrete frequencies to generate the Fourier Series coefficients $c_m = (1/T_1)X_{\mathbf{pulse}}(m\omega_1)$.

Our next task is to compute $|X(\omega)|^2$:

$$|X(\omega)|^2 = \{\sum_{m=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\; 2\pi\delta(\omega T_1 - 2\pi m)\}\;\{\sum_{n=-\infty}^{\infty} X_{\mathbf{pulse}}(n\omega_1)^*\; 2\pi\delta(\omega T_1 - 2\pi n)\}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\, X_{\mathbf{pulse}}(n\omega_1)^*\; 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(\omega T_1 - 2\pi n)\ .$$

Looking at the product of the two delta functions, there can be no contribution to the double sum unless m = n, so we continue

$$= \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2\; 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(\omega T_1 - 2\pi m)\ .$$

Since both delta functions have the same argument, the effect of one delta is to set the argument of the other to 0, so we continue

$$= \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2\; 2\pi\delta(0)\, 2\pi\delta(\omega T_1 - 2\pi m)$$

so that

$$| X(\omega) |^2 = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} | X_{\mathbf{pulse}}(m\omega_1) |^2 2\pi\delta(\omega T_1 - 2\pi m) .$$

As defined in (32.6) we then get this result for the spectral energy density of a pulse train,

$$\mathcal{E}(\omega) = \frac{|X(\omega)|^2}{2\pi} = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2 \ \delta(\omega T_1 - 2\pi m) \quad . \tag{33.1}$$

Since an infinite pulse train x(t) has infinite total energy, we are not surprised to find that the spectral energy density is also infinite due to the $\delta(0)$.

Appendix A shows that $[2\pi\delta(0)] = (2N+1)$ if we "undo the limit", allowing us to write the previous equation as (we will do this again below in section (b))

$$\frac{\mathcal{E}(\omega)}{(2N+1)} = \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2 \ \delta(\omega T_1 - 2\pi m) \tag{33.2}$$

The quantity on the left is the spectral energy density of an average pulse in the pulse train and is a sensible quantity even in the limit $N \to \infty$. If we divide this by the duration of a pulse T_1, we obtain the spectral *power* density of an average pulse in the pulse train. But this is just the average spectral power density of the pulse train taken as a whole, which we shall give the name $\mathcal{P}(\omega)$. Thus the spectral power density of a pulse train is given by

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{(2N+1)T_1} = \frac{|X(\omega)|^2}{2\pi(2N+1)T_1} = \frac{1}{T_1} \sum_{m=-\infty}^{\infty} |X_{\mathbf{pulse}}(m\omega_1)|^2 \ \delta(\omega T_1 - 2\pi m) \tag{33.3}$$

The underlying pulse has spectral energy density $|X_{\mathbf{pulse}}(m\omega_1)|^2/2\pi$ and duration T_1, so we could also define

$$\mathcal{P}_{\mathbf{pulse}}(\omega) \equiv \frac{\mathcal{E}_{\mathbf{pulse}}(\omega)}{T_1} = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} = \text{spectral power density of the underlying pulse} \tag{33.4}$$

We end up then with this expression for the spectral power density of an infinite pulse train :

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1) \, 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.5}$$

where the delta function is dimensionless because its argument is dimensionless. An equivalent form of the above is, using $\omega_1 = T_1/2\pi$,

$$\mathcal{P}(\omega) = \frac{1}{T_1} \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1) \, 2\pi\delta(\omega - m\omega_1) = \omega_1 \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1) \, \delta(\omega - m\omega_1) \tag{33.6}$$

The power spectrum is entirely discrete, being a series of δ lines separated by $\omega_1 = 2\pi/T_1$ whose amplitudes are controlled by the envelope function $\omega_1 \mathcal{P}_{pulse}(\omega)$.

(b) Finite Simple Pulse Train

Recall these equations from the end of Section 14,

$$x(t) = \sum_{n=-N}^{N} x_{pulse}(t - nT_1) \qquad (14.1)$$

$$X(\omega) = X_{pulse}(\omega)\, 2\pi\, \delta_5(\omega T_1, N) \qquad (14.4)$$

where δ_5 is a certain delta function model described in Appendix A. We then have

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2 \; [2\pi\, \delta_5(\omega T_1, N)]^2 \quad . \qquad (33.7)$$

Recalling the definition of the δ_6 delta function model from Appendix A,

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \; \frac{[2\pi\delta_5(\omega T_1, N)]^2}{(2N+1)} \qquad (A.20)$$

we obtain

$$\frac{1}{2\pi} \; \frac{|X(\omega)|^2}{(2N+1)} = |X_{pulse}(\omega)|^2 \;\; \delta_6(\omega T_1, N)$$

to which we add common factors on both sides to get

$$\frac{1}{2\pi} \frac{1}{T_1} \frac{|X(\omega)|^2}{(2N+1)} = \frac{\mathcal{E}(\omega)}{(2N+1)T_1} = \frac{1}{T_1} |X_{pulse}(\omega)|^2 \; \frac{1}{2\pi} \; 2\pi\, \delta_6(\omega T_1, N)$$

Using the definitions of (33.3) for $\mathcal{P}(\omega)$ and (33.4) for $\mathcal{P}_{pulse}(\omega)$, this becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, 2\pi\, \delta_6(\omega T_1, N) \qquad (33.8)$$

which is a purely continuous spectrum which has strong peaks separated by 2π when N is large.

For $N \to \infty$, we use result (A.21)

$$\lim_{N\to\infty} \delta_6(\omega T_1, N) = \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \qquad (A.21)$$

to obtain from (33.8)

$$\mathcal{P}(\omega) \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega)\, 2\pi\, \delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N) \;=\; \mathcal{P}_{\mathtt{pulse}}(\omega)\, 2\pi \sum_{m\,=\,-\infty}^{\infty} \delta(\omega T_{\mathtt{1}}-2\pi m)$$

$$= \sum_{m\,=\,-\infty}^{\infty} \mathcal{P}_{\mathtt{pulse}}(\omega)\, 2\pi\, \delta(\omega T_{\mathtt{1}}-2\pi m) \;=\; \sum_{m\,=\,-\infty}^{\infty} \mathcal{P}_{\mathtt{pulse}}(m\omega_{\mathtt{1}})\, 2\pi\, \delta(\omega T_{\mathtt{1}}-2\pi m)$$

which replicates (33.5) obtained for the infinite pulse train. This derivation avoids the unpleasant $\delta(0)$ object, but requires Appendix A information about the $\delta_{\mathtt{5}}$ and $\delta_{\mathtt{6}}$ delta function models. It is somewhat of a tradeoff.

Example: We run a 1 GHz pulse train into an HP Spectrum Analyzer and we assume that it takes 1 msec to compute each frequency bin. For a given bin, we have a finite pulse train of $N = 10^{\mathtt{6}}$ pulses. In (33.6) the width of one of the delta peaks in $\delta_{\mathtt{6}}(k,N)$ is $\Delta k \approx \pi/N$ (Appendix A) so $\Delta(\omega T_{\mathtt{1}}) = \pi/N$ and $\Delta\omega = \omega_{\mathtt{1}}/N$. If a spectral line falls in some bin, it will have the following line width

$$\Delta\omega = \omega_{\mathtt{1}}/\,N \qquad \text{or} \qquad \Delta f = f_{\mathtt{1}}/N \;\;.$$

In our example, the line width Δf would be $10^{\mathtt{9}}/10^{\mathtt{6}} = 1$ kHz. This is just an application of the uncertainty principle which says $\Delta t\, \Delta f \approx 1$. Here, Δt is the time of measurement (the 1 msec), and Δf is the line width (the 1 KHz). The frequency is uncertain by amount Δf, so the line has width Δf.

(c) Autocorrelation and Average Power for a Simple Pulse Train

We can compute the autocorrelation function for pulse train x(t) directly in the time domain using definition (32.1), and we shall do this later with the addition of statistics. Here instead we inverse Fourier transform both sides of (33.1),

$$\mathcal{E}(\omega) \;=\; \frac{|X(\omega)|^{\mathtt{2}}}{2\pi} \;=\; [2\pi\delta(0)] \sum_{m\,=\,-\infty}^{\infty} |X_{\mathtt{pulse}}(m\omega_{\mathtt{1}})|^{\mathtt{2}} \;\; \delta(\omega T_{\mathtt{1}} - 2\pi m). \qquad \textit{(33.1)}$$

Using (1.2) apply $\int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}$ to both sides to get

$$\int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}\, \frac{|X(\omega)|^{\mathtt{2}}}{2\pi} = [2\pi\delta(0)] \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} \sum_{m\,=\,-\infty}^{\infty} |X_{\mathtt{pulse}}(m\omega_{\mathtt{1}})|^{\mathtt{2}} \;\; \delta(\omega T_{\mathtt{1}} - 2\pi m)$$

From (32.8) we know that $|X(\omega)|^{\mathtt{2}} = R_{\mathtt{x}}(\omega)$, so from (1.2) the left side is just $r_{\mathtt{x}}(t)$. On the right side the delta function pins the phasor ω to a specific value and the result is

$$\frac{r_{\mathtt{x}}(t)}{2\pi\delta(0)} \;= (1/T_{\mathtt{1}}) \sum_{m\,=\,-\infty}^{\infty} |X_{\mathtt{pulse}}(m\omega_{\mathtt{1}})|^{\mathtt{2}}\, e^{im\omega_{\mathtt{1}}t} \qquad (33.9)$$

Undoing the limit,

$$\frac{r_x(t)}{(2N+1)} = (1/T_1) \sum_{m=-\infty}^{\infty} |X_{pulse}(m\omega_1)|^2 \, e^{im\omega_1 t} \tag{33.10}$$

In this undone limit, we define the duration of the pulse train to be

$$T \equiv (2N+1)T_1 \tag{33.11}$$

Dividing both sides of (33.10) by T_1 then gives

$$\frac{r_x(t)}{T} = (1/T_1)^2 \sum_{m=-\infty}^{\infty} |X_{pulse}(m\omega_1)|^2 \, e^{im\omega_1 t} \tag{33.12}$$

But we know that $(1/T_1)\, X_{pulse}(m\omega_1) = c(m\omega_1) = c_m$, the complex Fourier coefficient of (14.13). So we can rewrite the above result one more time to get a simple result for the autocorrelation function of x(t),

$$\frac{r_x(t)}{T} = \sum_{m=-\infty}^{\infty} |c_m|^2 \, e^{im\omega_1 t} \tag{33.13}$$

Finally, we can set t=0 to get,

$$\frac{r_x(0)}{T} = \sum_{m=-\infty}^{\infty} |c_m|^2 \tag{33.14}$$

Recall from (32.10) that $r_x(0)$ is the total energy in the pulse train, so $r_x(0)/T$ is the *average power* carried by the pulse train (delivered to a 1Ω load). We can then write

$$\langle p \rangle = \frac{1}{T} \int_{-\infty}^{\infty} dt\, p(t) = \text{average power} = \sum_{m=-\infty}^{\infty} |c_m|^2 \quad .$$

Some texts add (1/T) in their definition of $r_x(t)$, but we have chosen not to do that.

So, the average power in a pulse train is the sum of the squared magnitudes of the complex Fourier Series coefficients. We know that $c_{-m} = c_m^*$ when x(t) is real, so we can reflect the negative sum over to the positive side and gain a factor of 2. We also know that $2c_m = a_m - ib_m$ from (15.4), so this gives us several other ways to write the above sum

$$\begin{aligned} \langle p \rangle &= |c_0|^2 + 2 \sum_{m=1}^{\infty} |c_m|^2 \\ &= (1/4)\,|a_0|^2 + (1/2) \sum_{m=1}^{\infty} \{\, |a_m|^2 + |b_m|^2 \,\} \end{aligned} \tag{33.15}$$

The magnitude-squares of the Fourier Series coefficients give a power decomposition of a pulse train x(t).

34. Statistical Amplitude Modulated Pulse Trains Part I

Earlier we used w(t) for a pulse train, here it is called x(t), so $W(\omega) \equiv X(\omega)$.

(a) Infinite pulse trains

Suppose we have, in place of the infinite pulse train of (33.1) with all coefficients unity, a new pulse train where the coefficients are completely general numbers, presumably real. Later we might want to have each coefficient be either a 1 or a 0. For now, we stay general, so here is our amplitude modulated pulse train,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \, x_{\mathbf{pulse}}(t - nT_1) \quad . \tag{34.1}$$

Dimensions: Think of $x_{\mathbf{pulse}}(t)$ and x(t) as volts, so the a_n are dimensionless.

Now, lets imagine that we have a "statistical ensemble" of such pulse trains. Each such pulse train gets a label i. Then we can say,

$$x^{(i)}(t) = \sum_{n=-\infty}^{\infty} a_n{}^{(i)} x_{\mathbf{pulse}}(t - nT_1) \quad . \tag{34.2}$$

We now define the operation < > to indicate the average of some quantity over a large statistical ensemble of systems. Assume there are ' I ' systems in the ensemble. We then have,

$$\langle r_x(t) \rangle = (1/I) \sum_{i=1}^{I} \int_{-\infty}^{\infty} dt' \, x^{(i)}(t') \, x^{(i)}(t' + t) \quad . \tag{34.3}$$

If we now insert (34.2) twice into (33.3),

$$\langle r_x(t) \rangle = (1/I) \sum_{i=1}^{I} \int_{-\infty}^{\infty} dt' \, \{ \sum_{m=-\infty}^{\infty} a_m{}^{(i)} \, x_{\mathbf{pulse}}(t' - mT_1) \} \{ \sum_{n=-\infty}^{\infty} a_n{}^{(i)} \, x_{\mathbf{pulse}}(t' + t - nT_1) \}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{ (1/I) \sum_{i=1}^{I} a_m{}^{(i)} \, a_n{}^{(i)} \} \int_{-\infty}^{\infty} dt' \, x_{\mathbf{pulse}}(t' - mT_1) \, x_{\mathbf{pulse}}(t' + t - nT_1)$$

Following our statistical ensemble averaging rule, we define

$$\langle a_m a_n \rangle \equiv (1/I) \sum_{i=1}^{I} a_m{}^{(i)} a_n{}^{(i)} \tag{34.4}$$

so that

$$\langle r_{\mathbf{x}}(t)\rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle \int_{-\infty}^{\infty} dt'\; x_{\mathbf{pulse}}(t' - mT_1)\, x_{\mathbf{pulse}}(t' + t - nT_1) \tag{34.5}$$

Now replace integration variable t' by t'' = t' - mT_1 to evalaute the integral appearing above,

$$\int_{-\infty}^{\infty} dt''\, x_{\mathbf{pulse}}(t'')\, x_{\mathbf{pulse}}(t'' + \{\, t + (m-n)T_1 \,\}) \;= r_{\mathbf{pulse}}[t + (m-n)T_1] \tag{34.6}$$

and we end up with

$$\langle r_{\mathbf{x}}(t)\rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle\, r_{\mathbf{pulse}}[t + (m-n)T_1)] \tag{34.7}$$

This expresses the autocorrelation function of the pulse *train* as a sum, with statistically averaged coefficients, over the autocorrelation function of the pulse used to build the pulse train.

We wish to Fourier transform (34.7) to the ω domain using (1.1), so apply $\int_{-\infty}^{\infty} dt\, e^{-i\omega t}$ to both sides,

$$\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, \langle r_{\mathbf{x}}(t)\rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle \int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, r_{\mathbf{pulse}}[t + (m-n)T_1)]$$

On the left move the integration inside the statistical sum implied by <...> to get $\langle R_{\mathbf{x}}(\omega)\rangle$. On the right, replace integration variable t by t' = t + (m-n)T_1 so that $e^{-i\omega t} = e^{-i\omega t'}\, e^{i\omega(m-n)T_1}$. The result is

$$\langle R_{\mathbf{x}}(\omega)\rangle = R_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle\, e^{i\omega(m-n)T_1} \tag{34.8}$$

We now use the relation (32.8) to replace each R(ω) with it's corresponding spectral density. Thus,

$$\langle |X(\omega)|^2\rangle = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle\, e^{i\omega(m-n)T_1} \tag{34.9}$$

Divide both sides by 2π and use (33.1) and (33.4) to write this as

$$\langle\, \mathcal{E}(\omega)\,\rangle = \mathcal{E}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle a_m a_n\rangle\, e^{i\omega(m-n)T_1} \tag{34.10}$$

This result describes the spectral energy density of a statistical pulse train in terms of the spectral energy density of the pulse we select to build the train, and in terms of the statistically averaged coefficients. The integral $\int_{-\infty}^{\infty} d\omega\; \mathcal{E}_{\mathbf{pulse}}(\omega)$ is the (finite) total energy in one of the pulses from which the pulse train is

constructed, while $\int_{-\infty}^{\infty} d\omega < \mathcal{E}(\omega)>$ is the total energy in the average pulse train of a statistical ensemble of pulse trains. If $N = \infty$, then this integral is infinite.

If we now make some statistical assumptions about the coefficients, we will arrive at the spectrum of a statistical pulse train. This is exactly what we want to know.

We already see from (34.9) a principle fact. The spectral density of the pulse acts as an "envelope" function, regardless of what happens with the double summation business. More on this below.

(b) Finite pulse trains

Everything in the previous section is the same except we replace infinite sums by finite ones. Here are a few of the results :

$$x(t) = \sum_{n=-N}^{N} a_n \, x_{pulse}(t - nT_1) \ . \tag{34.11}$$

$$<r_x(t)> = \sum_{m=-N}^{N} \sum_{n=-N}^{N} <a_m a_n> r_{pulse}[t + (m-n)T_1)] \tag{34.12}$$

$$<|X(\omega)|^2> = |X_{pulse}(\omega)|^2 \sum_{m=-N}^{N} \sum_{n=-N}^{N} <a_m a_n> e^{i\omega(m-n)T_1} \tag{34.13}$$

$$< \mathcal{E}(\omega) > = \mathcal{E}_{pulse}(\omega) \sum_{m=-N}^{N} \sum_{n=-N}^{N} <a_m a_n> e^{i\omega(m-n)T_1} \tag{34.14}$$

35. Statistical Amplitude Modulated Pulse Trains Part II

(a) Statistics

By our definition (34.4) above, we have this average over our statistical ensemble of pulse *trains*:

$$\langle a_m a_n \rangle \equiv (1/I) \sum_{i=1}^{I} a_m^{(i)} a_n^{(i)} \quad . \tag{35.1}$$

A very simple case to consider is this:

$$\begin{aligned} a_m &= A \qquad \text{probability } p \\ a_m &= B \qquad \text{probability } 1\text{-}p \end{aligned} \tag{35.2}$$

Remember that index m labels pulse location m in the pulse train. If probabilities of a pulse value at locations m and at n are uncorrelated (they are independent of each other), then we can write the averaged coefficients as follows, first for two different locations $m \neq n$, then for the same location $m = n$:

$$\begin{aligned} n \neq m \qquad & \langle a_m a_n \rangle = [pp]\,AA + [p(1-p)]\,AB + [(1-p)p]BA + [(1-p)(1-p)]BB \equiv \alpha \\ n = m \qquad & \langle a_m^2 \rangle = [p]AA + [(1-p)]\,BB \equiv \beta \end{aligned} \tag{35.3}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. The reader must now stop reading and stare at the above equations until they make complete sense. Notice that the second line is quite distinct from the first line. In the double summation in (34.9), there are both "diagonal" terms where m = n, and off diagonal terms where m≠n. These groupings must be treated separately according to the above. We have defined new symbols α and β to emphasize that, for the situations shown, there is no longer any dependence on the n and m indices for these quantities. In a later section, we shall encounter a situation where the above equations do not apply because there is correlation between positions m and n (the AMI line code).

<u>Comment on expectation values, correlation and covariance</u>

In the language of statistics, one can imagine a "random variable" A_m associated with location m in our pulse train, and this variable can take certain values a_m (perhaps 0 and 1). We imagine a statistical ensemble of pulse trains indexed by i, and at location m the i_{th} pulse train has value $a_m^{(i)}$ of A_m.

If all pulse trains in our ensemble of pulse trains count the same in our average (they do), then one can write $\langle a_m a_n \rangle$ in (35.1) as

$$E(A_m A_n) = (1/I) \sum_{i=1}^{I} a_m^{(i)} a_n^{(i)} \quad \text{in the sense of} \quad E(XY) = (1/I) \sum_{i=1}^{I} x^{(i)} y^{(i)}$$

where E(q) means the expectation value of quantity q. This particular expectation value has a special name: it is called the **correlation** of X and Y. So in fact we have

$$\langle a_m a_n \rangle = E(A_m A_n) = (1/I) \sum_{i=1}^{I} a_m^{(i)} a_n^{(i)} \equiv \text{corr}(A_m A_n) \qquad \text{// correlation of } A_m \text{ and } A_n$$

If $\text{corr}(A_m A_n)$ is a significant positive or negative number, then the random variables A_m and A_n are strongly correlated in some way. This means that the probabilities of values of $a_m^{(i)}$ and $a_n^{(i)}$ are related in some way. One might wonder how there could be a correlation between locations m and n in a pulse train. We shall see this happen below in the discussion of the AMI lines code.

If A_m and A_n are independent random variables, we expect $\text{corr}(A_m A_n) \approx 0$.

Two other expectation values of interest are these, the **mean** and the **covariance** :

$$E(A_m) = (1/I) \sum_{i=1}^{I} a_m^{(i)} = \langle a_m \rangle \qquad \text{// mean of } A_m$$

$$E([A_m - \langle a_m \rangle][[A_n - \langle a_n \rangle]) = (1/I) \sum_{i=1}^{I} [a_m^{(i)} - \langle a_m \rangle] [a_n^{(i)} - \langle a_n \rangle]$$

$$= \text{cov}(A_m A_n) \qquad \text{// covariance of } A_m \text{ and } A_n$$

If it happens that $\langle a_m \rangle = 0$ for a certain pulse train, then covariance and correlation are the same.

(b) Infinite Pulse train with Random Coefficients

Using the above forms for the coefficient averages, (34.9) becomes

$$\langle |X(\omega)|^2 \rangle = |X_{\text{pulse}}(\omega)|^2 \{ \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} \qquad (35.4)$$

To evaluate the first double sum, we write it as

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [e^{i\omega(m-n)T_1}] = \sum_{n=-\infty}^{\infty} \{ \sum_{m=-\infty}^{\infty} [e^{i\omega(m-n)T_1}] - 1 \}$$

$$= (\sum_{n=-\infty}^{\infty} e^{+i\omega n T_1}) (\sum_{m=-\infty}^{\infty} e^{-i\omega m T_1}) - \sum_{n=-\infty}^{\infty} 1 = | \sum_{n=-\infty}^{\infty} e^{+i\omega n T_1} |^2 - \sum_{n=-\infty}^{\infty} 1$$

We now quote two results from Appendix A

$$\sum_{n=-\infty}^{\infty} 1 = [2\pi \delta(0)] \qquad \textit{(A.32)}$$

$$\{ \sum_{n=-\infty}^{\infty} e^{in\omega T_1} \}^2 = [\, 2\pi\delta(0)\,] \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \qquad (A.35)$$

so the first double sum in (35.4) becomes

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = [\, 2\pi\delta(0)\,] \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) - 1\} \qquad (35.5)$$

The second double sum in (35.4) is just

$$\sum_{n=-\infty}^{\infty} \sum_{m=n} [1] = \sum_{n=-\infty}^{\infty} [1] = [\, 2\pi\, \delta(0)] \qquad (35.6)$$

and therefore we may write (35.4) as

$$\langle |X(\omega)|^2 \rangle = |X_{pulse}(\omega)|^2 \{ \alpha\, [\, 2\pi\delta(0)\,] \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) - 1\} + \beta\, [\, 2\pi\, \delta(0)] \}$$

$$= [\, 2\pi\delta(0)\,]\, |X_{pulse}(\omega)|^2 \{ \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) - 1\} + \beta\}$$

so

$$\frac{\langle |X(\omega)|^2 \rangle}{2\pi\delta(0)} = |X_{pulse}(\omega)|^2\, [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\,] \qquad (35.7)$$

Next, we undo the limit replacing $2\pi\delta(0) = (2N+1)$ on the left. At the same time we divide both sides by $2\pi T_1$. To get

$$\frac{\langle |X(\omega)|^2 \rangle}{2\pi(2N+1)T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1}\, [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\,]$$

Then using (33.3) and (33.4) this becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega)\, [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\,] \qquad (35.8)$$

Here $\mathcal{P}_{pulse}(\omega)$ is the spectral power density of $x_{pulse}(t)$, and its integral over $d\omega$ is finite. The left side is $\langle \mathcal{P}(\omega) \rangle$ which is the statistically averaged spectral power density of the pulse train, and its integral over $d\omega$ is also finite.

(c) Finite Pulse train with Random Coefficients

We repeat the previous section for a finite random pulse train. Using the above forms for the coefficient averages, (34.13) becomes

$$\langle|X(\omega)|^2\rangle = |X_{\mathtt{pulse}}(\omega)|^2 \left\{ \alpha \sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] + \beta \sum_{n=-N}^{N} \sum_{m=n} [1] \right\} \tag{35.9}$$

To evaluate the first double sum, we write it as

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = \sum_{n=-N}^{N} \{ \sum_{m=-N}^{N} [\, e^{i\omega(m-n)T_1}] - 1 \}$$

$$= \Big(\sum_{n=-N}^{N} e^{+i\omega n T_1} \Big) \Big(\sum_{m=-N}^{N} e^{-i\omega m T_1} \Big) - \sum_{n=-N}^{N} 1 = \Big| \sum_{n=-N}^{N} e^{+i\omega n T_1} \Big|^2 - (2N+1)$$

From Appendix A we have

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} = 2\pi\delta_5(k,N)$$

so the first double sum in (35.9) becomes

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = [2\pi\delta_5(\omega T_1,N)]^2 - (2N+1) \quad . \tag{35.10}$$

The second double sum in (35.9) is just

$$\sum_{n=-N}^{N} \sum_{m=n} [1] = \sum_{n=-N}^{N} [1] = (2N+1)$$

and therefore we may write (35.9) as

$$\langle|X(\omega)|^2\rangle = |X_{\mathtt{pulse}}(\omega)|^2 \{ \alpha \, [\, [2\pi\delta_5(\omega T_1,N)]^2 - (2N+1) \,] + \beta \, (2N+1) \}$$

$$= (2N+1) \; |X_{\mathtt{pulse}}(\omega)|^2 \{ \alpha \, [\frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} - 1] + \beta \, (2N+1) \}$$

so

$$\frac{\langle|X(\omega)|^2\rangle}{(2N+1)} = |X_{\mathtt{pulse}}(\omega)|^2 \left\{ (\beta-\alpha) + \alpha \, \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \right\} \tag{35.11}$$

The factor multiplying α can be replaced using (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \qquad (A.20)$$

to give

$$\frac{<|X(\omega)|^2>}{(2N+1)} = |X_{\mathbf{pulse}}(\omega)|^2 \{ (\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1,N) \} \qquad (35.12)$$

Divide both sides by $2\pi T_1$ and use (33.3) and (33.4) to get

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega) [(\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1,N)] \qquad (35.13)$$

which is the finite pulse train version of (35.8). In the limit $N \to \infty$ we know from (A.21)

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.21)$$

so that (35.13) becomes

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega) [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(k-2\pi m)]$$

which reproduces (35.8).

(d) Statistical Pulse Trains Summary and Examples

Spectral Energy Density of a Random Pulse Train (35.14)

$$x(t) = \sum_{n=-\infty}^{\infty} a_n\, x_{\mathbf{pulse}}(t - nT_1) \quad (34.1) \qquad \text{or} \qquad \sum_{n=-N}^{N} a_n\, x_{\mathbf{pulse}}(t - nT_1) \quad (34.11)$$

$\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi(2N+1)T_1}$ = spectral power density for a pulse train, total power is $\int d\omega\, \mathcal{P}(\omega)$

$<\mathcal{P}(\omega)>$ = average spectral power density for an ensemble of pulse trains

$\mathcal{P}_{\mathbf{pulse}}(\omega) = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1}$ = spectral power density for the underlying pulse

Spectral energy density per pulse for a statistical pulse train (average of ensemble):

$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega) [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)]$ // infinite (35.8)

$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega) [(\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1,N)]$ // finite (35.13)

If $P(f)df = \mathcal{P}(\omega)d\omega = \mathcal{P}(\omega)\, 2\pi df$, then $\langle P(f)\rangle = 2\pi\mathcal{P}(\omega)$, in any of the lines above.

Alternate form for the above equations:

$$\frac{\langle|X(\omega)|^2\rangle}{2\pi\delta(0)} = |X_{\mathbf{pulse}}(\omega)|^2 \left[(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\right] \quad \text{// infinite} \qquad (35.7)$$

$$\frac{\langle|X(\omega)|^2\rangle}{(2N+1)} = |X_{\mathbf{pulse}}(\omega)|^2 \left[(\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1, N)\right] \quad \text{// finite} \qquad (35.12)$$

$\alpha = [pp]\, AA + [p(1-p)]\, AB + [(1-p)p]BA + [(1-p)(1-p)]BB$ // $= \langle a_m a_n\rangle$ when $n \neq m$ (35.3)

$\beta = [p]AA + [(1-p)]\, BB$ // $= \langle a_m a_n\rangle$ when $n=m$

A pulse has probably p of having amplitude A, and probability 1-p of having amplitude B.

A=1 and B=0 => $\alpha = p^2$ $\beta = p$ $(\beta - \alpha) = p(1-p)$

This is a fascinating result for $\langle\mathcal{P}(\omega)\rangle$ for an infinite pulse train. There are two weighted terms: the first term has weight $(\beta - \alpha)$, the second term has weight α. The first term is just $(\beta-\alpha)$ times the spectrum of the pulse used to build the pulse train -- it is a continuous function of ω. The second term is the same discrete spectrum we found in (33.3) for the simple pulse train. It's delta functions pick off specific values of $|X_{\mathbf{pulse}}(\omega)|^2$ at the lines.

Here are a few examples using A = 1 and B = 0.

Example 1: Let p=1 with A=1 and B=0. This is just the simple pulse train analyzed in Section 33 (a) for which we found that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1)\, 2\pi\delta(\omega T_1 - 2\pi m) \qquad (33.5)$$

Since p = 1, we find $\alpha = p^2 = 1$ and $\beta = p = 1$, so $(\beta-\alpha) = 0$ and our (35.8) reduces to the above.

Example 2: Let p=0 with A=1 and B=0. Then every pulse has B = 0, $x(t) \equiv 0$, $\alpha = \beta = 0$, and both terms in (38.5) vanish so there is zero spectral energy density.

Example 3: Let p=1/2 with A=1 and B=0, so $\alpha = p^2 = 1/4$ and $\beta = p = 1/2$, This pulse train then has an equal probably of 1's and 0's. We find from (38.5) and (35.13) that

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\right] \quad \text{infinite} \qquad (35.15)$$

$$\langle \mathscr{P}(\omega) \rangle = \mathscr{P}_{\mathtt{pulse}}(\omega)\,[(1/4) + (1/4)\,2\pi\,\delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N)\,] \qquad \text{finite} \qquad (35.16)$$

In (35.15) the discrete spectrum has been reduced to 1/4 of its full strength, and a continuous spectrum exists with coefficient 1/4 as shown.

(e) A detailed numerical statistical pulse train example

In this example, we shall average a small ensemble of pulse trains with p = 1/2 in an attempt to verify the result (35.16) stated above, which we rewrite as

$$\frac{\langle |X(\omega)|^{\mathtt{2}} \rangle}{(2N+1)} = |X_{\mathtt{pulse}}(\omega)|^{\mathtt{2}}\,[(1/4) + (1/4)\,2\pi\,\delta_{\mathtt{6}}(\omega T_{\mathtt{1}},N)\,] \quad . \qquad (35.17)$$

We shall use the standard square pulse of height 1 and $\tau = T_{\mathtt{1}} = 1$ so that, from (9.2),

$$|X(\omega)_{\mathtt{pulse}}| = \text{sinc}(\omega/2) \quad .$$

Since our pulse train will be fairly short (N = 20 pulses) and since we shall only average a small number of pulse trains (M = 10), we know our result will not exactly match (35.17). Still, we hope to see in our result some kind of continuous background spectrum which approximates the curve $(1/4)|X_{\mathtt{pulse}}(\omega)|^{\mathtt{2}} = (1/4)\,\text{sinc}^{\mathtt{2}}(\omega/2)$, and we expect to see a delta-function-like peak which, since $2\pi\delta_{\mathtt{6}}(0,N) = (2N+1)$, has a peak value of about $(1/4)41 = 10.25$. Since this will be added to the continuous background, the peak should have a height of $10.25 + .25 = 10.5$. However, for our small ensemble, we won't have exactly p = 1/2, so the delta peak won't be exactly 10.5 units high.

We know that $\delta_{\mathtt{6}}$ has identical peaks spaced by 2π, but we expect the non-central peaks to be suppressed by the $\text{sinc}^{\mathtt{2}}(\omega/2)$ zeros which occur at $\omega = n(2\pi)$.

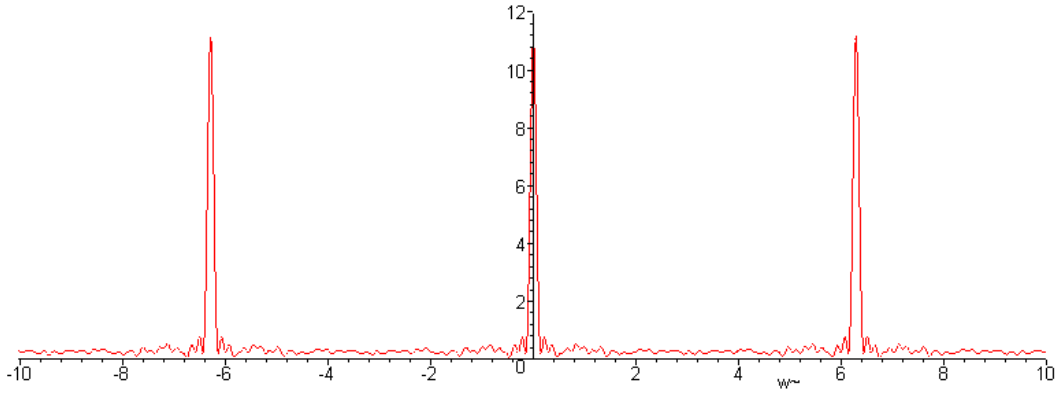
First, here the self-documented Maple program which generates $\langle |X(\omega)|^{\mathtt{2}} \rangle$. The program also generates the quantity $\langle X(\omega) \rangle$ upon which we shall comment in Section 36 below.

```
restart; assume(w,real); with(plots):
T1 is pulse duration, N is number of pulses in each pulse train
T1 := 1: N := 20:
Spectrum for standard box shaped pulse
Xpulse := T1*sin(w*T1/2)/(w*T1/2):
Subsequent calls to r() will generate either 0 or 1 with prob p = 1/2
r := rand(0..1):
```

```
Clear averaging accumulators. M is number of pulse trains to average
Xas means X absolute-value squared
X_acc := 0:Xas_acc := 0: M := 10:
for m from 1 to M do
  Create a random pulse train
  for n from -N to N do y[n] := r(); od:
  Compute the Digital Fourier Transform of the pulse train from (23.5)
  Yp := T1*sum(y[j]*exp(-I*w*j*T1),j=-N..N):
  Compute X(w) for the pulse train from (25.4), omitting Xpulse(w)
  X := (1/T1)* Yp;
  Accumulate both X(w) and |X(w)|^2, divide by M later for ensemble average
  X_acc := X_acc + X;
  Xas_acc := Xas_acc + eval(abs(X)^2);
od:
Compute <X(w)> and <|X(w)|^2> from the accumulators, but Xpulse not yet added
X_av_   :=  X_acc/M:
Xas_av_ := Xas_acc/M:
```

At this point, before $X_{\mathtt{pulse}}(\omega)$ is added to the result, we plot $\langle|X(\omega)|^2\rangle$. As expected, we see the peaks of δ_6 spaced by 2π and having height around 10 units,

```
Plot <|X(w)|^2>/(2N+1) without the pulse shape
plot(Xas_av_/(2*N+1), w=-10..10,color = [red,blue],view = 0..12,numpoints = 500);
```

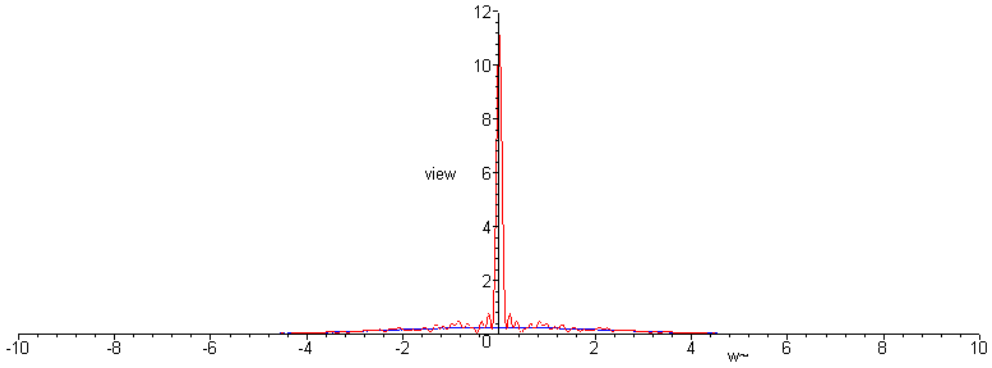


We now insert copies of $X_{\mathtt{pulse}}(\omega)$ as appropriate,

```
Install the pulse shape to get the true expressions for <X(w)> and of <|X(w)|^2>
X_av := Xpulse*X_av_:
Xas_av := Xpulse^2*Xas_av_:
```

and then we can plot $\dfrac{\langle|X(\omega)|^2\rangle}{(2N+1)}$ for ω in the same range (-10,10)

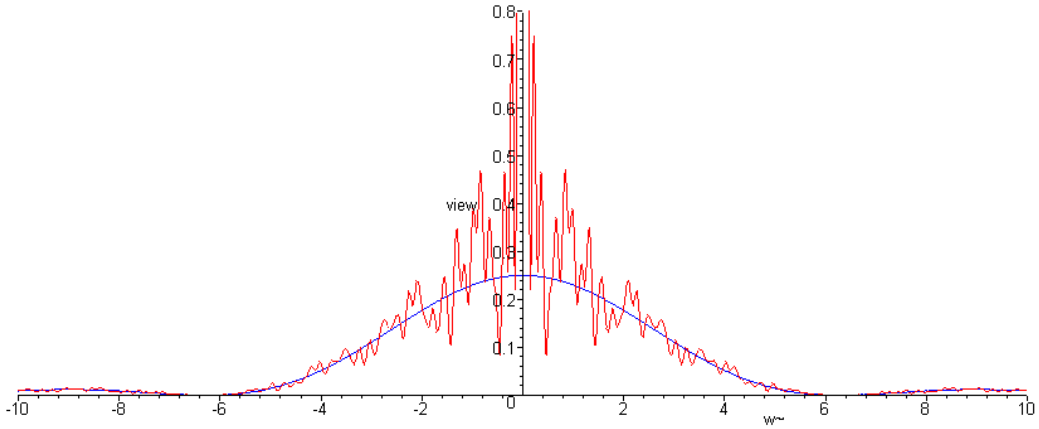
```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..12 ,color = [red,blue],numpoints = 500);
```



We see that the zeros of the sinc function have killed off the adjacent peaks.

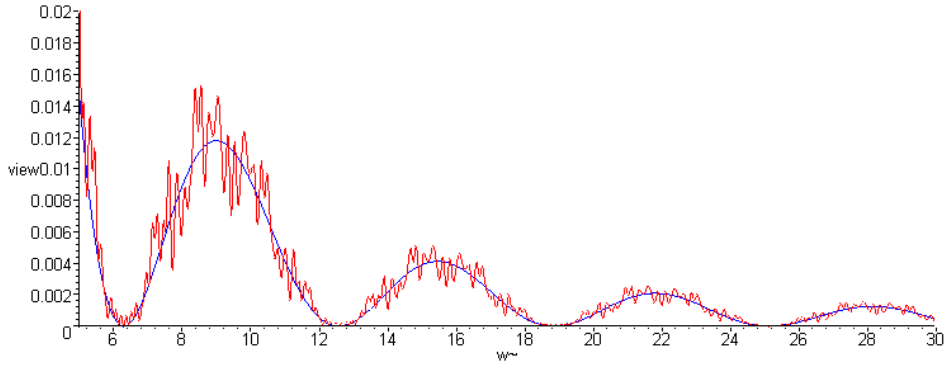
Next, we restrict the plot height to be 0.8 units to view the detail, chopping off the δ_6 peak,

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..0.8 ,color = [red,blue],numpoints = 500);
```



The spectrum $\frac{<|X(\omega)|^2>}{(2N+1)}$ is seen to have a continuous component which very well approximates one quarter of the sinc^2 curve, as we hoped it would. This tracking also occurs away from the central peak. Here is a blow-up of the above plot for ω in the range (5,30)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=5..30, view=0..0.02 ,color = [red,blue],numpoints = 500);
```



It might be noted that Maple does this work analytically, so that Was-av is a function of ω having a huge number of trigonometric terms. For the reader's interest, we show Was-av(ω) for a typical program run :

$$
\begin{aligned}
Was_av := \frac{2}{5}\sin\left(\frac{1}{2}w\right)^2 \Big(& (2\cos(8w) + 2\cos(10w) + \cos(9w) + \cos(12w) + 2\cos(11w) + \cos(14w) + \cos(17w) + 2\cos(5w) + \cos(4w) + 2\cos(3w) \\
& + \cos(w) + \cos(2w) + 2\cos(20w) + 2\cos(18w) + \cos(19w) + \cos(15w) + \cos(13w))^2 \\
& + (\sin(9w) + \sin(12w) + \sin(14w) + \sin(17w) - \sin(4w) + \sin(w) + \sin(2w) - \sin(19w) - \sin(15w) - \sin(13w))^2 + (1 + 2\cos(8w) + \cos(10w) \\
& + 2\cos(11w) + 2\cos(13w) + 2\cos(14w) + 2\cos(16w) + \cos(6w) + 2\cos(5w) + 2\cos(4w) + \cos(2w) + \cos(20w) + 2\cos(19w) + \cos(17w) + \cos(9w) \\
& + \cos(7w))^2 + (\sin(10w) - \sin(6w) - \sin(2w) - \sin(20w) - \sin(17w) - \sin(9w) - \sin(7w))^2 + (1 + 2\cos(7w) + \cos(9w) + \cos(13w) + \cos(14w) \\
& + 2\cos(15w) + \cos(8w) + 2\cos(6w) + 2\cos(5w) + \cos(4w) + 2\cos(3w) + 2\cos(2w) + 2\cos(w) + \cos(20w) + \cos(18w) + \cos(12w) + \cos(19w))^2 \\
& + (\sin(9w) + \sin(13w) + \sin(14w) - \sin(8w) - \sin(4w) - \sin(20w) - \sin(18w) - \sin(12w) + \sin(19w))^2 + (1 + 2\cos(10w) + \cos(13w) + 2\cos(14w) \\
& + 2\cos(15w) + 2\cos(16w) + \cos(17w) + \cos(8w) + 2\cos(5w) + \cos(2w) + 2\cos(w) + \cos(6w) + 2\cos(18w) + \cos(19w) + \cos(9w) + \cos(7w) \\
& + \cos(20w))^2 + (\sin(13w) + \sin(17w) - \sin(8w) - \sin(2w) + \sin(6w) - \sin(19w) - \sin(9w) - \sin(7w) + \sin(20w))^2 + (\cos(7w) + \cos(10w) + \cos(9w) \\
& + 2\cos(12w) + 2\cos(13w) + \cos(15w) + \cos(5w) + 2\cos(3w) + \cos(w) + \cos(4w) + \cos(6w) + 2\cos(19w) + \cos(11w) + \cos(20w))^2 \\
& + (\sin(7w) + \sin(10w) + \sin(9w) + \sin(15w) - \sin(5w) - \sin(w) + \sin(4w) + \sin(6w) - \sin(11w) + \sin(20w))^2 + (1 + 2\cos(7w) + \cos(12w) + \cos(13w) \\
& + \cos(14w) + 2\cos(15w) + \cos(16w) + \cos(8w) + 2\cos(5w) + 2\cos(3w) + 2\cos(w) + \cos(2w) + \cos(4w) + \cos(20w) + \cos(17w) + \cos(9w) \\
& + \cos(18w))^2 + (\sin(12w) + \sin(13w) + \sin(14w) + \sin(16w) - \sin(8w) + \sin(2w) + \sin(4w) - \sin(20w) - \sin(17w) - \sin(9w) + \sin(18w))^2 + (1 \\
& + 2\cos(7w) + \cos(12w) + 2\cos(11w) + \cos(14w) + 2\cos(17w) + \cos(8w) + 2\cos(6w) + \cos(5w) + \cos(4w) + 2\cos(3w) + \cos(2w) + \cos(w) \\
& + \cos(20w) + \cos(16w) + \cos(9w))^2 + (\sin(12w) + \sin(14w) - \sin(8w) - \sin(5w) - \sin(4w) - \sin(2w) + \sin(w) - \sin(20w) - \sin(16w) - \sin(9w))^2 + (1 \\
& + 2\cos(7w) + \cos(8w) + 2\cos(12w) + 2\cos(13w) + \cos(14w) + 2\cos(17w) + 2\cos(6w) + \cos(5w) + \cos(4w) + \cos(3w) + 2\cos(w) + 2\cos(18w) \\
& + \cos(11w) + \cos(10w))^2 + (\sin(8w) + \sin(14w) - \sin(5w) - \sin(4w) - \sin(3w) - \sin(11w) - \sin(10w))^2 \\
& + (\cos(8w) + 2\cos(10w) + \cos(13w) + 2\cos(6w) + \cos(5w) + \cos(w) + \cos(2w) + \cos(3w) + \cos(20w) + \cos(17w) + \cos(12w) + \cos(9w) + \cos(7w))^2 \\
& + (\sin(8w) + \sin(13w) - \sin(5w) - \sin(w) + \sin(2w) + \sin(3w) - \sin(20w) - \sin(17w) - \sin(12w) - \sin(9w) - \sin(7w))^2 + (\cos(8w) + \cos(10w) \\
& + 2\cos(14w) + \cos(16w) + \cos(17w) + \cos(6w) + \cos(5w) + 2\cos(4w) + \cos(3w) + \cos(w) + \cos(20w) + 2\cos(18w) + \cos(19w) + \cos(12w) \\
& + \cos(7w))^2 + (\sin(8w) + \sin(10w) + \sin(16w) + \sin(17w) - \sin(6w) - \sin(5w) - \sin(3w) + \sin(w) - \sin(20w) - \sin(19w) - \sin(12w) - \sin(7w))^2 \Big) \Big/ w^2
\end{aligned}
$$

In more serious work with larger numbers, one would of course do this in a more numeric fashion, but we are able to confirm the basic results even with this small experiment.

section 26 starting part

In this Chapter we introduced some new definitions, such as the autocorrelation function, and the Wiener-Khintchine relation. We used these in our derivation of the statistical average of the spectral density of a pulse train.

The whole thing was somewhat of a deception, and here we want to make sure the reader has not missed this point. Our main purpose for introducing autocorrelation and Wiener-Khintchine was to get these well-known terms into our dialog, since they appear in texts.

In reality, there is nothing at all new in this Chapter, and we certainly did not need to use the words autocorrelation or Wiener-Khintchine to get our main results.

As shown in (32.7), autocorrelation is just a standard convolution of x(t) with x(-t), and Wiener-Khintchine is just the convolution theorem (3.6) applied to this convolution equation, with (7.2) and (7.1) thrown in. In effect, we could have just defined R(ω) to be $|X(\omega)|^2$, given it the new name "autocorrelation", and then forgotten about it.

Here for example is a simple derivation of (34.10):

Start off with the pulse train "i" with statistical coefficients $a_n^{(i)}$, as in (34.2),

$$x^{(i)}(t) = \sum_{n=-\infty}^{\infty} a_n^{(i)} \, x_{pulse}(t - nT_1) \tag{36.1}$$

Fourier transform both sides in the usual manner, picking up the usual time-shift phase,

$$X^{(i)}(\omega) = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} a_n^{(i)} \, e^{-in\omega T_1} \tag{36.2}$$

Magnitude-square this thing to get

$$|X^{(i)}(\omega)|^2 = |X_{pulse}(\omega)|^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_m^{(i)} a_n^{(i)} \, e^{i\omega(m-n)T_1} \tag{36.3}$$

Finally, divide by 2π, use definition (33.1) for $\mathcal{E}$, and average over ensemble index i to get

$$< \mathcal{E}(\omega) > = \mathcal{E}_{pulse}(\omega) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} < a_m a_n > e^{i\omega(m-n)T_1}$$

which is (34.10).

The derivation of (35.8) would then proceed as outlined in Section 35 with no mention of the autocorrelation function.