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Chapter 6: Statistical Pulse Trains

32. The Autocorrelation Function

This is a very simple idea which is often made to seem complicated. Start with a reasonable function x(t). Define the autocorrelation function of x(t) as follows:

$$r_{\mathbf{x}}(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \qquad (32.1)$$

The integrand is the function evaluated at time t' times the same function evaluated at later time t'+t. Some texts might define the above with an extra overall "normalizing" constant factor; we leave it as shown.

A simple reflection property follows from the above definition (use t" = t' + t):

$$r_{\mathbf{x}}(-t) = r_{\mathbf{x}}(t) \qquad (32.2)$$

Thus, in (32.1) we could put either +t or -t in the last parentheses.

Although we have not yet mentioned statistics and randomness, one could easily imagine the following situation. Suppose x(t') is some sort of random function ("noise") that takes values in the range -1 to 1. It seems likely that for a value of the separation t that is larger than some small value, you might get $r_{\mathbf{x}}(t) = 0$. The vague argument would be that there is no "correlation" between x(t') and x(t'+t), so the product of these two functions ought to be pretty random, and a sum of random numbers in the range -1 to 1 ought to be zero. Even in this case, we can see that the result is not zero if t = 0, since we are then summing a positive quantity. In fact, $r_{\mathbf{x}}(0)$ is the area under $x(t)^2$.

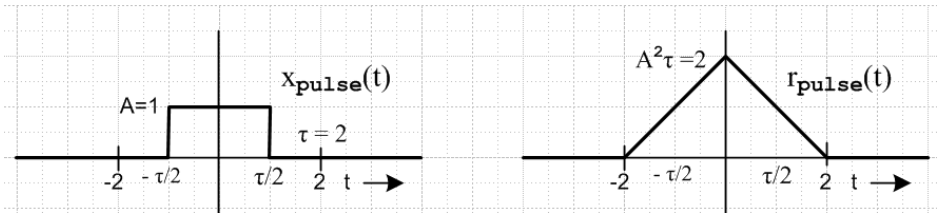
(a) Autocorrelation function for a Square Pulse.

Before going any further, let us compute the autocorrelation function for some simple case we are familiar with. A good candidate is x(t) = a square pulse of width τ and amplitude A. As in (9.1),

$$x_{\mathbf{pulse}}(t) = A\,[\,\theta(t + \tau/2) - \theta(t - \tau/2)\,] \quad . \qquad \textit{(9.1)}$$

One can easily do the above integral (32.1) to get the answer, but it is very obvious what the answer is. We are multiplying a box times a box shifted by t. Where they overlap, the integrand is A^2. The boxes only overlap if the absolute value of shift t is less than the width τ of the pulse. If this is so, the size of the overlap is τ-|t|. If |t| is larger than τ, there is no overlap, so the integral is 0. Thus,

$$r_{\mathbf{pulse}}(t) = A^2(\tau-|t|)\,\theta(\tau-|t|) = (A^2\tau)\,[\,1 - |t|/\tau\,]\,\theta(\tau-|t|) \qquad (32.3)$$



(b) Energy in a finite signal x(t)

The total energy in a finite duration signal x(t) can be computed in either the t-domain or the ω-domain using Parseval's formula (10.5), to which we add 1/R to each side.

$$E = \int_{-\infty}^{\infty} dt\, |x(t)|^2/R = \int_{-\infty}^{\infty} d\omega\, \frac{|X(\omega)|^2}{2\pi R} \qquad (32.4)$$

If we think of x(t) as the voltage across a resistor R, then dt $x^2(t)/R$ is the energy delivered to the resistor in time dt, and the integral on the left is the total energy in signal x(t). The dimensions on the right are, looking at (1.1),

$$d\omega\, \frac{|X(\omega)|^2}{2\pi R} = \sec^{-1}\,(\text{volt-sec})^2/\text{ohms} = (\text{volt}^2/\text{ohms})\text{sec} = \text{watts-sec} = \text{joules} = \text{energy} \qquad (32.5)$$

Setting R = 1Ω, we can write this as

$$E = \int_{-\infty}^{\infty} dt\, p(t) = \int_{-\infty}^{\infty} d\omega\, \mathcal{E}(\omega) \qquad (32.6)$$

$p(t) \equiv |x(t)|^2$ = energy density in the t-domain	(joule/sec = watt)
p(t)dt = energy in dt	(joules)
$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain	(joule-sec)
$\mathcal{E}(\omega)d\omega$ = energy in dω	(joules)

We shall refer to $\mathcal{E}(\omega)$ as the **spectral energy density** of signal x(t) whose Fourier Transform is X(ω). The isolated single pulse $x_{pulse}(t)$ has a corresponding $\mathcal{E}_{pulse}(\omega)$.

(c) The Spectral Density Connection: Wiener-Khintchine

Changing to t" = -t', we can trivially rewrite the definition (32.1) as follows:

$$r_x(t) = \int_{-\infty}^{\infty} dt''x(t - t'')\, x(-t'') \quad . \qquad \text{// energy units} \qquad (32.7)$$

This has the standard convolution equation form (3.1) where we select

$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$

$c(t) = x(-t) \quad \leftrightarrow \quad C(\omega) = X(-\omega) = X(\omega)^* \qquad$ // from (7.1) and (7.2)

Thus, the diagonalized frequency domain form (3.6) is

$$R_{\mathbf{x}}(\omega) = |X(\omega)|^2 \tag{32.8}$$

Dividing by 2π we find that

$$\mathcal{E}(\omega) = (1/2\pi)\, R_{\mathbf{x}}(\omega) \tag{32.9}$$

This is the key result. It says that the Fourier integral transform of the autocorrelation function of x(t) is 2π times the spectral energy density of x(t). This seemingly simple result gets the elaborate label of a Wiener-Khintchine Relation, see Bennett and Davey, *Data Transmission*, p 334.

Note also from (32.1) that $r_{\mathbf{x}}(t)$ evaluated at t = 0 gives the total energy in signal x(t),

$$r_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} dt\, x^2(t) \equiv \text{ E = total energy in signal x(t)} \tag{32.10}$$

For us, the significance of (32.9) is that we can "inject statistics" into a computation of the autocorrelation function, and then we will know the power spectrum of our statistical signal from (32.9). All we have to do is Fourier transform the autocorrelation function $r_{\mathbf{x}}(t)$. Examples will follow.

(d) Verification of Wiener-Khintchine for a Square Pulse.

In (32.3) we have the autocorrelation function for a square pulse. One can insert this into the Fourier transform (1.1) to compute $R_{\mathbf{x}}(\omega)$.

$$R_{\mathbf{x}}(\omega) = \int_{-\tau}^{\tau} dt\, (A^2\tau)\, [\, 1 - |\, t\, |\, /\, \tau\,]\, e^{-i\omega t} = 2(A^2\tau) \int_{0}^{\tau} dt\, (1-t/\tau)\cos(\omega t)$$

$$= 2(A^2\tau)\,(1-\cos(\omega\tau))/(\omega^2\tau) = 4(A^2\tau)\sin^2(\omega\tau/2)/(\omega^2\tau) = (A^2\tau^2)\sin^2(\omega\tau/2)/(\omega\tau/2)^2$$

$$= (A\tau)^2\, [\, \mathrm{sinc}(\omega\tau/2)\,]^2 \tag{32.11}$$

and this is recognized from (9.2) to be $|X(\omega)|^2$ for the square pulse, in agreement with (32.8) .

(e) Cross-correlation, convolution, and autocorrelation

Note: a* means complex conjugation, $b * c$ means convolution, $b \star c$ means cross-correlation

The **cross-correlation** of two functions b and c is defined this way

$$(b \star c)(t) = \int_{-\infty}^{\infty} dt'\, b^*(t')c(t+t') = (c \star b)^*(-t) \tag{32.12}$$

where the right equality is easy to show setting t+t' = t''. So in general, $b \star c \neq c \star b$.

In Section 7 we noted that a function is Hermitian if f*(-t) = f(t).

Fact: If b and c are both Hermitian, then $b \star c = c \star b$. (32.13)

Proof: $b \star c = \int_{-\infty}^{\infty} dt'\, b^*(t')c(t+t') = \int_{-\infty}^{\infty} dt'\, b(-t')c^*(-t-t') = \int_{-\infty}^{\infty} dt'\, b(t''+t)c^*(t'') = c \star b$

If b and c are both real and both even, then again, $b \star c = c \star b$.

The **convolution** of two functions b and c we saw from the (3.1) and (3.2) was this

$$(b * c)(t) = \int_{-\infty}^{\infty} dt'\, b(t')c(t-t') = (c * b)(t)$$

In order to relate these two operations, we need to show more detail in the notation. Thus, using t'' = -t',

$$[b(t) \star c(t)](t) = \int_{-\infty}^{\infty} dt'\, b^*(t')c(t+t') = \int_{-\infty}^{\infty} dt''\, b^*(-t'')c(t-t'') = [b^*(-t) * c(t)](t)$$

Then if b(t) is a Hermitian function so b*(-t) = b(t), then we have shown that :

Fact: If b is Hermitian, then $b \star c = c * b$. (32.14)

If b = c = real, then we have from (32.1) and (32.2),

$$r_b(t) = \int_{-\infty}^{\infty} dt'\, b(t')b(t'+t) = \int_{-\infty}^{\infty} dt'\, b(t')b(t'-t)$$

so we have just proven this fact :

Fact: If b is real, then $r_b = b \star b = b * b$ (32.15)

Thus, for a real function b, the autocorrelation function is the cross-correlation function of b with itself. This then gives some motivation for the name associated with r_b. It is also the convolution of b with itself.

33. Energy and Power of a Pulse Train

In subsections (a) through (d) we deal only with simple pulse trains. Along the way, certain facts are developed which apply to general as well as simple pulse trains. In subsection (e) we gather together these general facts, and then show how quantities of interest can be related to the autocorrelation function.

(a) Infinite Simple Pulse Train

This is the first section in which we use the δ(0) notation of Appendix A which may make the reader feel a bit uncomfortable. In subsection (b) we shall repeat everything for a finite pulse train and then take the limit N→∞ to obtain the same results without using δ(0). In both sections we shall include the Z transform in passing, but our main work is in the ω variable, not the z variable.

From Section 14 (a) we know the spectrum of an infinite pulse train formed from pulses $x_{\mathbf{pulse}}(t)$ separated by time T_1 ,

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\mathbf{pulse}}(t - nT_1) \qquad (14.1)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m) \qquad (14.4)$$

We can obtain the same expressions from box (25.4) which summarizes amplitude modulated pulse trains by setting all amplitudes to $y_n = 1$ and changing w,W to x,X:

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\ x_{\mathbf{pulse}}(t - t_n) = \sum_{n=-\infty}^{\infty} x_{\mathbf{pulse}}(t - t_n) \qquad (33.1)$$

$$X(\omega) = (1/T_1)X_{\mathbf{pulse}}(\omega)\ X'(\omega) = X_{\mathbf{pulse}}(\omega)\ X''(z) \qquad (33.2)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} \quad . \qquad (33.3)$$

X'(ω) is the Digital Fourier Transform of x(t), and X''(z) is the Z Transform, where $z = e^{i\omega T_1}$

In the last line we then use (13.2)

$$\sum_{n=-\infty}^{\infty} e^{\pm ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

so that (33.3) becomes

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad (33.4)$$

and then (33.2) says

$$X(\omega) \;=\; (1/T_1)X_{\mathbf{pulse}}(\omega)\, X'(\omega) \;=\; X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.5}$$

in agreement with (14.4) quoted just above (33.1).

To find the frequency domain power spectrum of a signal x(t), our first task is to compute $|\, X(\omega)\,|^2$. From (33.2) we get

$$|X(\omega)|^2 \;=\; |X_{\mathbf{pulse}}(\omega)|^2\, (1/T_1)^2\, |X'(\omega)|^2 \;=\; |X_{\mathbf{pulse}}(\omega)|^2\, |\, X''(z)\,|^2 \qquad z = e^{i\omega T_1} \tag{33.6}$$

We therefore must deal with the following object, using (33.4),

$$|\, X''(z)\,|^2 \;=\; (1/T_1)^2\, |X'(\omega)|^2 \;=\; \Big[\; \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \;\Big]^2 \quad , \tag{33.7}$$

and we are now faced with the issue of squaring delta functions. Formally these objects don't exist in the realm of distribution theory, but as discussed in Appendix A, we can deal with them in an *ad hoc* way which proves to be useful.

$$\Big[\; \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \;\Big]^2 \;=\; \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(\omega T_1 - 2\pi n)$$

Looking at the product of the two delta functions, there can be no contribution to the double sum unless m = n, so we continue

$$= \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(0) \;=\; [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

so that

$$\Big[\; \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \;\Big]^2 \;=\; [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \tag{33.8}$$

The object $\delta(0)$ is formally undefined, but in Appendix A we ascribe the meaning that $2\pi\delta(0) = 2N+1$ in the limit that $N \to \infty$ and we can always "undo the limit" when necessary. We shall firm up this idea in section (b) directly below. So we have shown then that

$$|\, X''(z)\,|^2 \;=\; (1/T_1)^2\, |X'(\omega)|^2 \;=\; [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.9}$$

or

$$\frac{|X''(z)|^2}{[2\pi\delta(0)]} = (1/T_1)^2\frac{|X'(\omega)|^2}{[2\pi\delta(0)]} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.10}$$

Then from (33.6)

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 \frac{|X''(z)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 (1/T_1)^2\frac{|X'(\omega)|^2}{[2\pi\delta(0)]}$$

$$= |X_{pulse}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.11}$$

We shall now repeat the above set of steps for a finite pulse train.

(b) Finite Simple Pulse Train

Our finite pulse train always has pulses ranging from n = -N to N instead of from n = -∞ to ∞. We start off exactly as in the previous section but with limited sums

$$x(t) = \sum_{n=-N}^{N} y_n\ x_{pulse}(t - t_n) = \sum_{n=-N}^{N} x_{pulse}(t - t_n) \tag{33.12}$$

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\ X'(\omega) = X_{pulse}(\omega)\ X''(z) \tag{33.13}$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} y_n\, e^{-i\omega n T_1} = \sum_{n=-N}^{N} e^{-i\omega n T_1} \tag{33.14}$$

In the last line we then use (13.3)

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\ \delta_5(k,N) \qquad -\infty < k < \infty \qquad \textit{(13.3)}$$

where δ_5 is a periodic delta function model discussed in Appendix A (b). Equation (33.14) becomes

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi\ \delta_5(\omega T_1, N) \tag{33.15}$$

and then equation (33.13) says

$$X(\omega) = X_{pulse}(\omega)\ 2\pi\ \delta_5(\omega T_1, N)\ . \tag{33.16}$$

This δ_5 is periodic with period 2π and has identical peaks separated by 2π. For finite N, these are peaks of finite width and height. Squaring, we find

$$| X(\omega) |^2 = | X_{\mathbf{pulse}}(\omega) |^2 \; [2\pi \, \delta_5(\omega T_1,N)]^2 \quad . \tag{33.17}$$

Recalling the definition of the δ_6 delta function model from Appendix A,

$$2\pi \, \delta_6(k,N) \equiv \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} \tag{A.20}$$

we obtain

$$\frac{|X(\omega)|^2}{(2N+1)} = | X_{\mathbf{pulse}}(\omega) |^2 \; 2\pi \, \delta_6(\omega T_1,N) \tag{33.18}$$

and this is the finite pulse train result. We can then take the limit $N \to \infty$ and make use of

$$\lim_{N\to\infty} \delta_6(\omega T_1,N) = \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \tag{A.21}$$

to find that

$$\lim_{N\to\infty} \left[\frac{|X(\omega)|^2}{(2N+1)}\right] = | X_{\mathbf{pulse}}(\omega) |^2 \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \tag{33.19}$$

and this replicates (33.11) with the promised connection $2\pi\delta(0) = \lim_{N\to\infty} (2N+1)$.

It is useful now to provide some side-by-side comparisons of results

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.4)$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi \, \delta_5(\omega T_1,N) \qquad \text{finite} \qquad (33.15)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.5)$$

$$X(\omega) = X_{\mathbf{pulse}}(\omega) \; 2\pi \, \delta_5(\omega T_1,N) \; . \qquad \text{finite} \qquad (33.16)$$

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.11)$$

$$\frac{|X(\omega)|^2}{(2N+1)} = | X_{\mathbf{pulse}}(\omega) |^2 \; 2\pi \, \delta_6(\omega T_1,N) \qquad \text{finite} \qquad (33.18)$$

One can interpret $\frac{|X(\omega)|^2}{[2\pi\delta(0)]}$ as the value of $|X(\omega)|^2$ *per pulse* in an infinite pulse train.

(c) Spectral Energy/Power Density of a Simple Pulse Train

In this section, everything is in the frequency domain, nothing is in the time domain.

Recall from (32.6) that

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$\mathcal{E}(\omega)d\omega$ = energy in dω (joules)

where $\mathcal{E}(\omega)$ is the spectral energy density of signal x(t) whose Fourier Transform is X(ω).

If we divide $\mathcal{E}(\omega)$ by 2N+1 or $2\pi\delta(0)$ we obtain the pulse train's average spectral energy density per pulse, which is the same as the energy density of an average pulse in the pulse train. Using our two expressions above for infinite and finite pulse trains, we then find

$$\frac{\mathcal{E}(\omega)}{[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$\frac{\mathcal{E}(\omega)}{(2N+1)} = \frac{|X(\omega)|^2}{2\pi(2N+1)} = \frac{1}{2\pi}\,|\,X_{\mathtt{pulse}}(\omega)\,|^2\;\; 2\pi\,\delta_{\mathtt{6}}(\omega T_1, N) \qquad (33.20)$$

If we divide the energy of the average pulse by T_1, we obtain power of an average pulse, and this is the same as the average power of the pulse train. Thus,

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_1[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi T_1[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_1) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_1[2N+1]} = \frac{|X(\omega)|^2}{2\pi T_1[2N+1]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_1)\;\; 2\pi\,\delta_{\mathtt{6}}(\omega T_1, N) \qquad (33.21)$$

If is perhaps helpful to define

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad (33.22)$$

Then the above maybe be restated

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_1) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2\,(1/T_1)\;\; 2\pi\,\delta_{\mathtt{6}}(\omega T_1, N) \qquad (33.23)$$

Meanwhile, our $x_{pulse}(t)$ which lasts only for duration T_1 itself has an energy and a power

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \tag{33.24}$$

We can then write (***) in the following compact form

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules}$$

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\ 2\pi\ \delta_6(\omega T_1, N) \qquad \text{joules} \tag{33.25}$$

Notice that $\delta(\omega T_1 - 2\pi m)$ and $\delta_6(\omega T_1, N)$ are both dimensionless, so the dimensions in each equation trivially match. A power density $\mathcal{P}(\omega)$ has dimensions of energy = joules, so that $\mathcal{P}(\omega)d\omega$ then has the dimensions of joules/sec = watts, and this is the pulse train power contained in interval $d\omega$ of the spectrum.

For the infinite pulse train, we are always allowed to write

$$2\pi\ \delta(\omega T_1 - 2\pi m) = (2\pi/T_1)\ \delta(\omega - m(2\pi/T_1)) = \omega_1\ \delta(\omega - m\omega_1) \qquad \omega_1 \equiv 2\pi/T_1$$

to get

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \qquad \text{infinite} \tag{33.26}$$

which shows more explicitly that the power lines occur at the harmonics $\omega = m\omega_1$. Recall now these two earlier facts

$$c(\omega) \equiv (1/T_1)X_{pulse}(\omega) \tag{14.14}$$

$$c_m \equiv c(m\omega_1) = (1/T_1)X_{pulse}(m\omega_1) \tag{14.10}$$

For the infinite pulse train we can then write

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1}\ \frac{2\pi}{T_1} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)$$

$$= \frac{|X_{pulse}(\omega)|^2}{T_1^2} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = |c(\omega)|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} |c(m\omega_1)|^2\ \delta(\omega - m\omega_1)$$

so that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\ \delta(\omega - m\omega_1). \tag{33.27}$$

Recall from box (15.12) that Dim(c_m) = Dim[x(t)] = volts (say), so $|c_m|^2$ = watts into a 1Ω resistor, and since $\delta(\omega - m\omega_1)$ has dimensions sec, $|c_m|^2\ \delta(\omega - m\omega_1)$ then has dimensions watt-sec = joules, as befits any $\mathcal{P}(\omega)$ object.

If we assume x(t) is a real pulse train, then X(ω) is Hermitian $X(-\omega) = [\ X(\omega)]^*$ by (7.1), which means

$$c_{-m} = (1/T_1)X_{pulse}(-m\omega) = (1/T_1)[X_{pulse}(m\omega)]^* = c_m^*$$

so

$$|c_{-m}|^2 = |c_m|^2 \qquad x(t) \text{ real} \tag{33.28}$$

and then we can fold the negative part of the sum in (**) to get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\ \delta(\omega - m\omega_1) = |c_0|^2\,\delta(\omega) + 2\sum_{m=1}^{\infty} |c_m|^2\ \delta(\omega - m\omega_1) \tag{33.29}$$

and then finally, recalling from our Fourier Series box (15.2) that $c_m = [\ a_m - ib_m\ ]/2$ and $b_0 = 0$ we get

$$\mathcal{P}(\omega) = a_0^2\,\delta(\omega) + (1/2)\sum_{m=1}^{\infty} (a_m^2+b_m^2)\ \delta(\omega - m\omega_1) \tag{33.30}$$

(d) Average Power P for a Simple Pulse Train

We seek an expression for the average power P in a general pulse train. This is of course a time domain quantity, not a frequency domain quantity.

P = [total energy in pulse train / time duration of pulse train] = average pulse train power

$$= (1/T)\int_{-\infty}^{\infty} dt\ |x(t)|^2 = \int_{-\infty}^{\infty} d\omega\ \frac{|X(\omega)|^2}{2\pi T} \qquad \text{// from (32.4) with } R = 1\Omega$$

so

$$P = \int_{-\infty}^{\infty} d\omega\ \mathcal{P}(\omega)\ . \qquad \text{// from (33.23)} \tag{33.31}$$

This is certainly reasonable since $\mathcal{P}(\omega)$ is the average spectral power density of the pulse train.

For the special case of a simple pulse train, we found above that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\ \delta(\omega - m\omega_1) = a_0^2\,\delta(\omega) + (1/2)\sum_{m=1}^{\infty} (a_m^2+b_m^2)\ \delta(\omega - m\omega_1)$$

Therefore the power in a simple pulse train is given by

$$P = \sum_{m=-\infty}^{\infty} |c_m|^2 = a_0^2 + (1/2) \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \qquad (33.32)$$

(e) General Pulse Train results and connection with the Autocorrelation Function

Certain results of the previous section apply to general pulse trains and we gather them here:

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad \textit{(33.22)}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad \textit{(33.23)}$$

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \qquad \textit{(33.24)}$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) \quad . \qquad \textit{(33.31)}$$

We now bring the autocorrelation function into the discussion. Let x(t) be an arbitrary but real pulse train, and recall that

$$r_x(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \qquad \textit{(32.1)}$$

Then, using T from (33.22),

$$r_x(0) \equiv \int_{-\infty}^{\infty} dt'\, x(t')^2 = P\,T = E = \text{total energy in the pulse train} \qquad (33.33)$$

so the average pulse train power maybe written in terms of the autocorrelation function evaluated at t = 0,

$$P = r_x(0)/T \quad . \qquad (33.34)$$

We diagonalized (32.1) treated as a convolution equation to obtain

$$|X(\omega)|^2 = R_x(\omega)$$

which we called the Wiener-Khintchine Relation. Here $R_x(\omega)$ is the Fourier Integral Transform of the autocorrelation function $r_x(t)$ of x(t). Then from (33.23) that $\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T}$ we get

$$\mathcal{P}(\omega) = R_{\mathbf{x}}(\omega)/ (2\pi T) \tag{33.35}$$

In this way, both P and $\mathcal{P}(\omega)$ can be expressed in terms of the autocorrelation function. Thus, one approach to finding P and $\mathcal{P}(\omega)$ for a pulse train is to try and determine $r_{\mathbf{x}}(t)$.

Here then is a box summarizing all the general pulse train results:

Energy and Power Properties of a General Pulse Train (33.36)

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \quad \textit{(33.22)}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = R_{\mathbf{x}}(\omega)/ (2\pi T) \quad \text{joules} \quad \textit{(33.23) and (33.35)}$$

$$\mathcal{P}_{\mathbf{pulse}}(\omega) \equiv \frac{\mathcal{E}_{\mathbf{pulse}}(\omega)}{T_1} = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \quad \textit{(33.24)}$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) = r_{\mathbf{x}}(0)/T \quad \text{watts} \quad \textit{(33.31) and (33.34)}$$

If $P(f)df = \mathcal{P}(\omega)d\omega = \mathcal{P}(\omega)\, 2\pi df$, then $P(f) = 2\pi\mathcal{P}(\omega)$.

34. Statistical Amplitude Modulated Pulse Trains Part I

(a) Spectral Energy/Power Density for a General Pulse Train

In the previous Section we dealt with simple pulse trains. Here we consider the more general amplitude modulated pulse train. In all equations, one can replace $\sum_{n=-\infty}^{\infty}$ by $\sum_{n=-N}^{N}$ to adapt the equation to a finite pulse train instead of an infinite one.

$$x(t) = \sum_{n=-\infty}^{\infty} y_{\mathbf{n}}\, x_{\mathbf{pulse}}(t - t_{\mathbf{n}}) \tag{34.1}$$

$$X(\omega) = (1/T_1)X_{\mathbf{pulse}}(\omega)\, X'(\omega) = X_{\mathbf{pulse}}(\omega)\, X''(z) \tag{34.2}$$

$$X''(z) = X'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_{\mathbf{n}}\, e^{-i\omega n T_1}. \tag{34.3}$$

To find the frequency-domain power spectrum of a signal x(t), our first task is to compute $|X(\omega)|^2$. From (34.2) we get

$$|X(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 (1/T_1)^2 |X'(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 |X''(z)|^2 \qquad z = e^{i\omega T_1} \qquad (34.4)$$

We therefore must deal with the following object, using (33.4),

$$|X''(z)|^2 = (1/T_1)^2 |X'(\omega)|^2 = \Big| \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \Big|^2$$

$$= \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \sum_{m=-\infty}^{\infty} y_m^* e^{+i\omega m T_1} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.5)$$

so that

$$|X(\omega)|^2 = |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.6)$$

From box (33.36) we then have

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = (1/2\pi) |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.7)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = (1/2\pi T) |X_{\mathbf{pulse}}(\omega)|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.8)$$

which we can write as

$$\mathcal{E}(\omega) = T_1 \mathcal{P}_{\mathbf{pulse}}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.9)$$

$$\mathcal{P}(\omega) = T_1 \mathcal{P}_{\mathbf{pulse}}(\omega) (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega (m-n) T_1} \qquad (34.10)$$

$$P = \int_{-\infty}^{\infty} d\omega \, \mathcal{P}(\omega) \quad . \qquad \textit{(33.31)} \qquad (34.11)$$

Not knowing details of the y_n there is not much else we can do in these expressions.

(b) Spectral Energy/Power Density for a Statistical Pulse Train

We imagine now a large ensemble of I pulse trains in which a particular pulse train is labeled by index i. This pulse train has coefficients $y_n^{(i)}$. We are interested in the ensemble averages of $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$ and P. We indicate the ensemble average of some quantity Q as <Q>. By averaging the last three equations above we obtain

$$\langle \mathcal{E}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \tag{34.12}$$

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^* \, y_n \rangle \, e^{i\omega(m-n)T_1} \tag{34.13}$$

$$\langle P \rangle = \int_{-\infty}^{\infty} d\omega \, \langle \mathcal{P}(\omega) \rangle \quad . \tag{34.14}$$

where

$$\langle y_m^* \, y_n \rangle = (1/I) \langle y_m^{(i)*} \, y_n^{(i)} \rangle \tag{34.15}$$

Up to this point we have tried to be general, allowing the $y_m^{(i)}$ to be complex coefficients, but from now on we consider them to be real, so we can delete the asterisks in the above equations.

A very simple case to consider is this, where A and B are real,

$$\begin{aligned} y_m &= A \qquad \text{probability } p \\ y_m &= B \qquad \text{probability } 1\text{-}p \end{aligned} \tag{34.16}$$

Remember that index m labels pulse location m in the pulse train. If probabilities of a pulse value at locations m and at n are uncorrelated (they are independent of each other), then we can write the averaged coefficients as follows, first for two different locations $m \neq n$, then for the same location $m = n$:

$$\begin{aligned} n \neq m \qquad & \alpha \equiv \langle y_m y_n \rangle = [pp]\,AA + [p(1-p)]\,AB + [(1-p)p]BA + [(1-p)(1-p)]BB \\ n = m \qquad & \beta \equiv \langle y_m^2 \rangle = [p]AA + [(1-p)]\,BB \end{aligned} \tag{34.17}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. The reader must now stop reading and stare at the above equations until they make complete sense. Notice that the second line is quite distinct from the first line. In the double summation in (34.13), there are both "diagonal" terms where m = n, and off diagonal terms where m≠n. These groupings must be treated separately according to the above. We have defined new symbols α and β to emphasize that, for the situations shown, there is no longer any dependence on the n and m indices for these quantities. In a later section, we shall encounter a situation where the above equations do not apply because there *is* correlation between positions m and n (the AMI line code).

Inserting (34.17) into (34.13) gives

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \left\{ \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \right\} \tag{34.18}$$

In Section 35 we shall evaluate this important result for infinite and finite pulse trains, but we pause momentarily for a digression.

(c) Digression on expectation values, correlation and covariance

In the language of statistics, one can imagine a "random variable" Y_m associated with location m in our pulse train, and this variable can take certain values y_m (perhaps 0 and 1). We imagine a statistical ensemble of pulse trains indexed by i, and at location m the i_{th} pulse train has value $y_m^{(i)}$ of Y_m.

If all pulse trains in our ensemble of pulse trains count the same in our average (they do), then one can write $\langle y_m y_n \rangle$ in (35.1) as

$$E(Y_m Y_n) = (1/I) \sum_{i=1}^{I} y_m^{(i)} y_n^{(i)} \quad \text{in the sense of} \quad E(XY) = (1/I) \sum_{i=1}^{I} x^{(i)} y^{(i)}$$

where E(q) means the expectation value of quantity q. This particular expectation value has a special name: it is called the **correlation** of X and Y. So in fact we have

$$\langle y_m y_n \rangle = E(Y_m Y_n) = (1/I) \sum_{i=1}^{I} y_m^{(i)} y_n^{(i)} \equiv \text{corr}(Y_m Y_n) \qquad \text{// correlation of } Y_m \text{ and } Y_n$$

If $\text{corr}(Y_m Y_n)$ is a significant positive or negative number, then the random variables Y_m and Y_n are strongly correlated in some way. This means that the probabilities of values of $y_m^{(i)}$ and $y_n^{(i)}$ are related in some way. One might wonder how there could be a correlation between locations m and n in a pulse train. We shall see this happen below in the discussion of the AMI lines code.

If Y_m and Y_n are independent random variables, we expect $\text{corr}(Y_m Y_n) \approx 0$.

Two other expectation values of interest are these, the **mean** and the **covariance** :

$$E(Y_m) = (1/I) \sum_{i=1}^{I} y_m^{(i)} = \langle y_m \rangle \qquad \text{// mean of } Y_m$$

$$E([Y_m - \langle y_m \rangle][[Y_n - \langle y_n \rangle]) = (1/I) \sum_{i=1}^{I} [y_m^{(i)} - \langle y_m \rangle] [y_n^{(i)} - \langle y_n \rangle]$$

$$= \text{cov}(Y_m Y_n) \qquad \text{// covariance of } Y_m \text{ and } Y_n$$

If it happens that $\langle y_m \rangle = 0$ for a certain pulse train, then covariance and correlation are the same.

35. Statistical Amplitude Modulated Pulse Trains Part II

(a) Infinite Pulse train with Random Coefficients

Recall now equation (34.18) which we give a new number,

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \, \{ \; \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \; \} \tag{35.1}$$

To evaluate the first double sum, we write it as

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = \sum_{n=-\infty}^{\infty} \{ \sum_{m=-\infty}^{\infty} [\, e^{i\omega(m-n)T_1}] - 1 \}$$

$$= (\sum_{n=-\infty}^{\infty} e^{+i\omega n T_1}) (\sum_{m=-\infty}^{\infty} e^{-i\omega m T_1}) - \sum_{n=-\infty}^{\infty} 1 = | \sum_{n=-\infty}^{\infty} e^{+i\omega n T_1} |^2 - \sum_{n=-\infty}^{\infty} 1$$

We now quote two results from Appendix A

$$\sum_{n=-\infty}^{\infty} 1 = [\, 2\pi \, \delta(0)] \tag{A.32}$$

$$\{ \sum_{n=-\infty}^{\infty} e^{in\omega T_1} \}^2 = [\, 2\pi\delta(0) \,] \{ \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) \} \tag{A.35}$$

so the first double sum in (35.4) becomes

$$\sum_{n=-\infty}^{\infty} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = [\, 2\pi\delta(0) \,] \{ \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) - 1 \} \tag{35.5}$$

The second double sum in (35.4) is just

$$\sum_{n=-\infty}^{\infty} \sum_{m=n} [1] = \sum_{n=-\infty}^{\infty} [1] = [\, 2\pi \, \delta(0)] \tag{35.6}$$

and therefore we may write (35.1) as

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \; \{ \; \alpha \, [\, 2\pi\delta(0) \,] \, [\sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) - 1] + \beta \, [\, 2\pi \, \delta(0)] \; \}$$

$$= \mathcal{P}_{pulse}(\omega) \, (T_1 \, [\, 2\pi\delta(0) \,]/T) \, \{ \, \alpha \, [\sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) - 1] + \beta \, \}$$

or

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \, \{ \, (\beta - \alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega T_1 - 2\pi m) \, \} \tag{35.7}$$

where we used $T = [2\pi\delta(0)]T_1$ from (33.22).

(b) Finite Pulse train with Random Coefficients

Recall again equation (34.18) but assume now a finite pulse train so the sums are different

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{\mathbf{pulse}}(\omega) \, (1/T) \, \{ \; \alpha \sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] \; + \beta \sum_{n=-N}^{N} \sum_{m=n} [1] \; \} \tag{35.8}$$

To evaluate the first double sum, we write it as

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = \sum_{n=-N}^{N} \{ \sum_{m=-N}^{N} [\, e^{i\omega(m-n)T_1}] - 1 \; \}$$

$$= (\sum_{n=-N}^{N} e^{+i\omega n T_1} \,) (\sum_{m=-N}^{N} e^{-i\omega m T_1} \,) - \sum_{n=-N}^{N} 1 \quad = | \sum_{n=-N}^{N} e^{+i\omega n T_1} |^2 \quad - (2N+1) \tag{35.9}$$

From Appendix A we have

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \, \{ \, \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \, \} \quad = 2\pi \delta_5(k,N) \tag{A.**}$$

so the first double sum in (35.9) becomes

$$\sum_{n=-N}^{N} \sum_{m \neq n} [\, e^{i\omega(m-n)T_1}] = \quad [2\pi\delta_5(\omega T_1,N)]^2 \; - (2N+1) \quad . \tag{35.10}$$

The second double sum in (35.9) is just

$$\sum_{n=-N}^{N} \sum_{m=n} [1] = \sum_{n=-N}^{N} [1] = (2N+1) \tag{35.11}$$

and therefore we may write (35.8) as

$$\langle \mathcal{P}(\omega) \rangle = T_1 \, \mathcal{P}_{\mathbf{pulse}}(\omega) \, (1/T) \, \{ \; \alpha \, \{ \, [2\pi\delta_5(\omega T_1,N)]^2 \; - (2N+1) \} + \beta(2N+1) \; \}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \, ((2N+1) \, T_1/T) \, \{ \; \alpha \, \{ \, \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \; - 1) \} + \beta \; \}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega) \, \{ \; (\beta-\alpha) \; + \alpha \, \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} \; \}$$

where this time we used $T = (2N+1)T_1$ from (33.22). The factor multiplying α can be replaced using (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(\omega T_1,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \qquad (A.20)$$

to give

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \{ \ (\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1,N) \ \} \qquad (35.12)$$

which is the finite pulse train version of (35.7). In the limit $N \to \infty$ we know from (A.21)

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.21)$$

so that (35.13) becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1-2\pi m) \}$$

which reproduces (35.7).

(c) Statistical Pulse Trains Summary and Examples

Here then is a brief summary of the above results:

Spectral Energy Density of a Random Pulse Train (35.13)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\, x_{pulse}(t - nT_1) \qquad \text{// or } \sum_{n=-N}^{N} \text{ for a finite pulse train}$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) [(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1- 2\pi m)] \qquad \text{// infinite} \qquad (35.7)$$

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) [(\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1,N)] \qquad \text{// finite} \qquad (35.12)$$

$$\alpha = [pp]\, AA + [p(1-p)]\, AB + [(1-p)p]BA + [(1-p)(1-p)]BB \quad \text{// } = \langle a_m a_n \rangle \text{ when } n \neq m$$

$$\beta = [p]AA + [(1-p)]\, BB \qquad \text{// } = \langle a_m a_n \rangle \text{ when } n=m \qquad (34.17)$$

A pulse has probably p of having amplitude A, and probability 1-p of having amplitude B.

A=1 and B=0 $\Rightarrow$ $\alpha = p^2$ $\quad \beta = p$ $\quad (\beta - \alpha) = p(1-p)$

Example 1: If we set p = 1 in the above box, we get $\alpha = 1$ and $(\beta-\alpha) = 0$. In this limit, our statistical average pulse train spectrum becomes the same as that for the simple pulse train of (33.25)

$$\mathscr{P}(\omega) \equiv \mathscr{P}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad \text{joules}$$

$$\mathscr{P}(\omega) \equiv \mathscr{P}_{\mathbf{pulse}}(\omega)\ 2\pi\ \delta_6(\omega T_1, N) \quad \text{joules} \qquad \textit{(33.25)}$$

This is because when p = 1 every pulse in the pulse train has the same amplitude A, which is how a simple pulse train is defined. As we reduce p below p = 1, the line spectra are scaled down by $\alpha = p^2 < 1$, and a continuous spectrum starts to appear with $(\beta - \alpha) = p(1-p)$. The randomness of the ensemble creates a continuous component in the spectral power density $\mathscr{P}(\omega)$.

Example 2: Let p=0 with A=1 and B=0. Then every pulse has B = 0, $x(t) \equiv 0$, $\alpha = \beta = 0$, and both terms in $\mathscr{P}(\omega)$ vanish so there is zero spectral energy density.

Example 3: Let p=1/2 with A=1 and B=0, so $\alpha = p^2 = 1/4$ and $\beta = p = 1/2$. This pulse train then has an equal probably of 1's and 0's. We find from box (35.13) that

$$\langle\mathscr{P}(\omega)\rangle = \mathscr{P}_{\mathbf{pulse}}(\omega)\ [(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m)\ ] \quad \text{infinite} \qquad (35.14)$$

$$\langle\mathscr{P}(\omega)\rangle = \mathscr{P}_{\mathbf{pulse}}(\omega)\ [(1/4) + (1/4)\ 2\pi\ \delta_6(\omega T_1, N)\ ] \quad \text{finite} \qquad (35.15)$$

In (35.15) the discrete spectrum has been reduced to 1/4 of its full strength, and a continuous spectrum exists with coefficient 1/4 as shown.

(e) A detailed numerical statistical pulse train example

Recall from box (33.36) that for a finite pulse train,

$$\mathscr{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T} \qquad \mathscr{P}_{\mathbf{pulse}}(\omega) = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \qquad T = (2N+1)T_1. \qquad (35.16)$$

Therefore we can write our p = 1/2 Example 3 result (35.15) as

$$\frac{\langle|X(\omega)|^2\rangle}{(2N+1)} = |X_{\mathbf{pulse}}(\omega)|^2\ [(1/4) + (1/4)\ 2\pi\ \delta_6(\omega T_1, N)\ ] \quad . \qquad (35.17)$$

We shall use for $x_{\mathbf{pulse}}(t)$ a square pulse of height 1 and $\tau = T_1 = 1$ so that, from (9.2),

$$|X_{\mathbf{pulse}}(\omega)| = \text{sinc}(\omega/2) \quad .$$

Since our pulse train will be fairly short (N = 20 pulses) and since we shall only average a small number of pulse trains (M = 10), we know our result will not exactly match (35.17). Still, we hope to see in our result some kind of continuous background spectrum which approximates the curve $(1/4)|X_{\mathbf{pulse}}(\omega)|^2 = (1/4)\ \text{sinc}^2(\omega/2)$, and we expect to see a delta-function-like peak which, since $2\pi\delta_6(0,N) = (2N+1)$, has a peak value of about $(1/4)41 = 10.25$. Since this will be added to the continuous background, the peak

should have a height of 10.25 + .25 = 10.5. However, for our small ensemble, we won't have exactly p = 1/2, so the delta peak won't be exactly 10.5 units high.

We know that δ_6 has identical peaks spaced by 2π, but we expect the non-central peaks to be suppressed by the $\text{sinc}^2(\omega/2)$ zeros which occur at $\omega = n(2\pi)$.

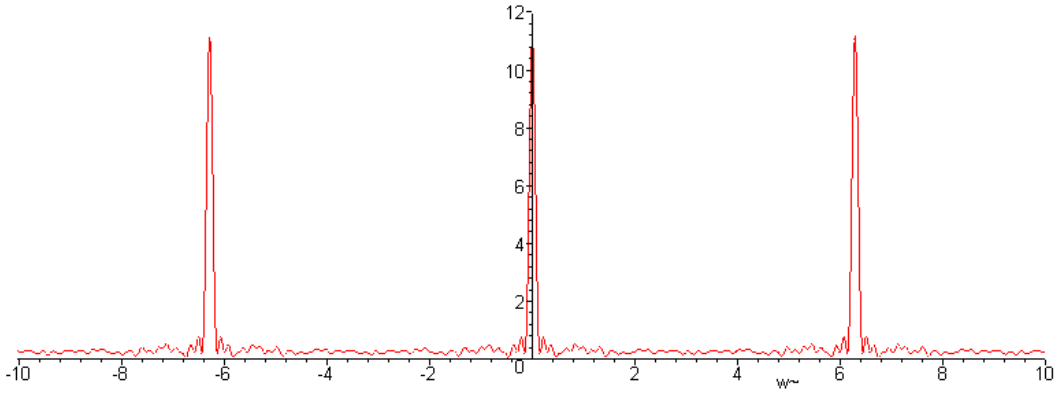
First, here the self-documented Maple program which generates $\langle|X(\omega)|^2\rangle$. The program also generates the quantity $\langle X(\omega)\rangle$ upon which we shall comment in Section 36 below.

```
restart; assume(w,real); with(plots):
T1 is pulse duration, N is number of pulses in each pulse train
T1 := 1: N := 20:
Spectrum for standard box shaped pulse
Xpulse := T1*sin(w*T1/2)/(w*T1/2):
Subsequent calls to r() will generate either 0 or 1 with prob p = 1/2
r := rand(0..1):
Clear averaging accumulators. M is number of pulse trains to average
Xas means X absolute-value squared
X_acc := 0:Xas_acc := 0: M := 10:
for m from 1 to M do
   Create a random pulse train
   for n from -N to N do y[n] := r(); od:
   Compute the Digital Fourier Transform of the pulse train from (23.5)
   Yp := T1*sum(y[j]*exp(-I*w*j*T1),j=-N..N):
   Compute X(w) for the pulse train from (25.4), omitting Xpulse(w)
   X := (1/T1)* Yp;
   Accumulate both X(w) and |X(w)|^2, divide by M later for ensemble average
   X_acc := X_acc + X;
   Xas_acc := Xas_acc + eval(abs(X)^2);
od:
Compute <X(w)> and <|X(w)|^2> from the accumulators, but Xpulse not yet added
X_av_    :=  X_acc/M:
Xas_av_ := Xas_acc/M:
```

At this point, before $X_{\texttt{pulse}}(\omega)$ is added to the result, we plot $\langle|X(\omega)|^2\rangle$. As expected, we see the peaks of δ_6 spaced by 2π and having height around 10 units,

Plot <|X(w)|^2>/(2N+1) without the pulse shape

```
plot(Xas_av_/(2*N+1), w=-10..10,color = [red,blue],view = 0..12,numpoints = 500);
```



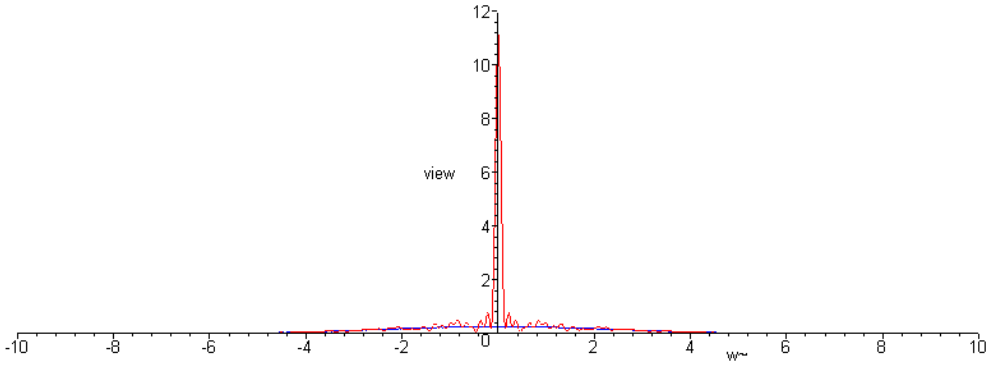
We now insert copies of $X_{\mathtt{pulse}}(\omega)$ as appropriate,

Install the pulse shape to get the true expressions for <X(w)> and of <|X(w)|^2>

```
X_av := Xpulse*X_av_:
Xas_av := Xpulse^2*Xas_av_:
```

and then we can plot $\frac{\langle|X(\omega)|^2\rangle}{(2N+1)}$ for ω in the same range (-10,10)

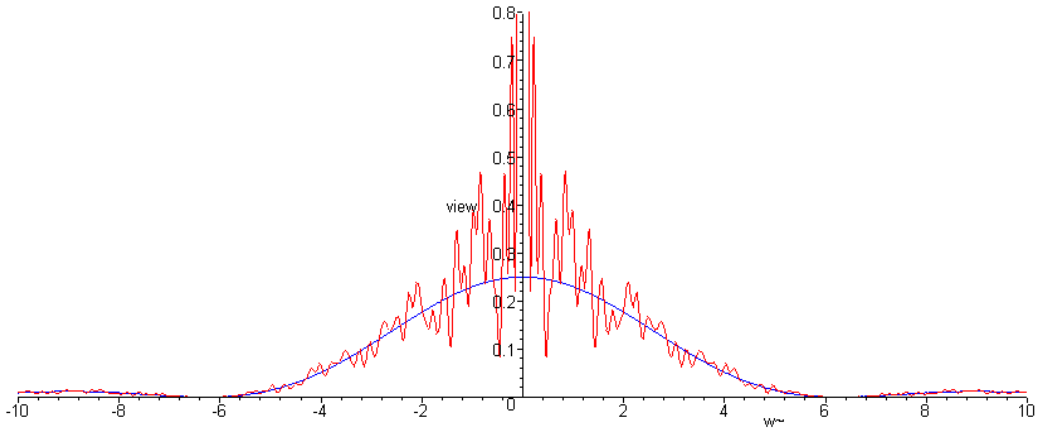
```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..12 ,color = [red,blue],numpoints = 500);
```



We see that the zeros of the sinc function have killed off the adjacent peaks.

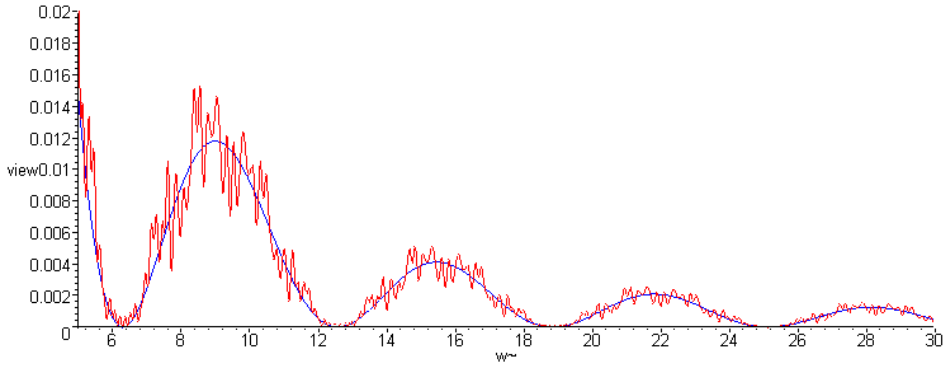
Next, we restrict the plot height to be 0.8 units to view the detail, chopping off the δ_6 peak,

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..0.8 ,color = [red,blue],numpoints = 500);
```



The spectrum $\frac{<|X(\omega)|^2>}{(2N+1)}$ is seen to have a continuous component which very well approximates one quarter of the sinc^2 curve, as we hoped it would. This tracking also occurs away from the central peak. Here is a blow-up of the above plot for ω in the range (5,30)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=5..30, view=0..0.02 ,color = [red,blue],numpoints = 500);
```



It might be noted that Maple does this work analytically, so that Was-av is a function of ω having a huge number of trigonometric terms. For the reader's interest, we show Was-av(ω) for a typical program run :

$$Was_av := \frac{2}{5}\sin\left(\frac{1}{2}w\right)^2 ((2\cos(8w) + 2\cos(10w) + \cos(9w) + \cos(12w) + 2\cos(11w) + \cos(14w) + \cos(17w) + 2\cos(5w) + \cos(4w) + 2\cos(3w)$$
$$+ \cos(w) + \cos(2w) + 2\cos(20w) + 2\cos(18w) + \cos(19w) + \cos(15w) + \cos(13w))^2$$
$$+ (\sin(9w) + \sin(12w) + \sin(14w) + \sin(17w) - \sin(4w) + \sin(w) + \sin(2w) - \sin(19w) - \sin(15w) - \sin(13w))^2 + (1 + 2\cos(8w) + \cos(10w)$$
$$+ 2\cos(11w) + 2\cos(13w) + 2\cos(14w) + 2\cos(16w) + \cos(6w) + 2\cos(5w) + 2\cos(4w) + \cos(2w) + \cos(20w) + 2\cos(19w) + \cos(17w) + \cos(9w)$$
$$+ \cos(7w))^2 + (\sin(10w) - \sin(6w) - \sin(2w) - \sin(20w) - \sin(17w) - \sin(9w) - \sin(7w))^2 + (1 + 2\cos(7w) + \cos(9w) + \cos(13w) + \cos(14w)$$
$$+ 2\cos(15w) + \cos(8w) + 2\cos(6w) + 2\cos(5w) + \cos(4w) + 2\cos(3w) + 2\cos(2w) + 2\cos(w) + \cos(20w) + \cos(18w) + \cos(12w) + \cos(19w))^2$$
$$+ (\sin(9w) + \sin(13w) + \sin(14w) - \sin(8w) - \sin(4w) - \sin(20w) - \sin(18w) - \sin(12w) + \sin(19w))^2 + (1 + 2\cos(10w) + \cos(13w) + 2\cos(14w)$$
$$+ 2\cos(15w) + 2\cos(16w) + \cos(17w) + \cos(8w) + 2\cos(5w) + \cos(2w) + 2\cos(w) + \cos(6w) + 2\cos(18w) + \cos(19w) + \cos(9w) + \cos(7w)$$
$$+ \cos(20w))^2 + (\sin(13w) + \sin(17w) - \sin(8w) - \sin(2w) + \sin(6w) - \sin(19w) - \sin(9w) - \sin(7w) + \sin(20w))^2 + (\cos(7w) + \cos(10w) + \cos(9w)$$
$$+ 2\cos(12w) + 2\cos(13w) + \cos(15w) + \cos(5w) + 2\cos(3w) + \cos(w) + \cos(4w) + \cos(6w) + 2\cos(19w) + \cos(11w) + \cos(20w))^2$$
$$+ (\sin(7w) + \sin(10w) + \sin(9w) + \sin(15w) - \sin(5w) - \sin(w) + \sin(4w) + \sin(6w) - \sin(11w) + \sin(20w))^2 + (1 + 2\cos(7w) + \cos(12w) + \cos(13w)$$
$$+ \cos(14w) + 2\cos(15w) + \cos(16w) + \cos(8w) + 2\cos(5w) + 2\cos(3w) + 2\cos(w) + \cos(2w) + \cos(4w) + \cos(20w) + \cos(17w) + \cos(9w)$$
$$+ \cos(18w))^2 + (\sin(12w) + \sin(13w) + \sin(14w) + \sin(16w) - \sin(8w) + \sin(2w) + \sin(4w) - \sin(20w) - \sin(17w) - \sin(9w) + \sin(18w))^2 + (1$$
$$+ 2\cos(7w) + \cos(12w) + 2\cos(11w) + \cos(14w) + 2\cos(17w) + \cos(8w) + 2\cos(6w) + \cos(5w) + \cos(4w) + 2\cos(3w) + \cos(2w) + \cos(w)$$
$$+ \cos(20w) + \cos(16w) + \cos(9w))^2 + (\sin(12w) + \sin(14w) - \sin(8w) - \sin(5w) - \sin(4w) - \sin(2w) + \sin(w) - \sin(20w) - \sin(16w) - \sin(9w))^2 + (1$$
$$+ 2\cos(7w) + \cos(8w) + 2\cos(12w) + 2\cos(13w) + \cos(14w) + 2\cos(17w) + 2\cos(6w) + \cos(5w) + \cos(4w) + \cos(3w) + 2\cos(w) + 2\cos(18w)$$
$$+ \cos(11w) + \cos(10w))^2 + (\sin(8w) + \sin(14w) - \sin(5w) - \sin(4w) - \sin(3w) - \sin(11w) - \sin(10w))^2$$
$$+ (\cos(8w) + 2\cos(10w) + \cos(13w) + 2\cos(6w) + \cos(5w) + \cos(w) + \cos(2w) + \cos(3w) + \cos(20w) + \cos(17w) + \cos(12w) + \cos(9w) + \cos(7w))^2$$
$$+ (\sin(8w) + \sin(13w) - \sin(5w) - \sin(w) + \sin(2w) + \sin(3w) - \sin(20w) - \sin(17w) - \sin(12w) - \sin(9w) - \sin(7w))^2 + (\cos(8w) + \cos(10w)$$
$$+ 2\cos(14w) + \cos(16w) + \cos(17w) + \cos(6w) + \cos(5w) + 2\cos(4w) + \cos(3w) + \cos(w) + \cos(20w) + 2\cos(18w) + \cos(19w) + \cos(12w)$$
$$+ \cos(7w))^2 + (\sin(8w) + \sin(10w) + \sin(16w) + \sin(17w) - \sin(6w) - \sin(5w) - \sin(3w) + \sin(w) - \sin(20w) - \sin(19w) - \sin(12w) - \sin(7w))^2) \Big/ w^2$$

In more serious work with larger numbers, one would of course do this in a more numeric fashion, but we are able to confirm the basic results even with this small experiment.

36. A View from the High Ground

(a) What role has the autocorrelation function played in our development?

In Section 32 the autocorrelation function was first introduced, as $r_x(t)$ for x(t), and was computed for the simple case of a box pulse. Treating the definition of $r_x(t)$ as a convolution equation, the Wiener-Khintchine Relation $R_x(\omega) = |X(\omega)|^2$ was trivially derived. It was then shown that the spectral energy density of a pulse train $\mathcal{E}(\omega) = (1/2\pi)\, R_x(\omega)$ due to this relation. Finally, it was shown that $r_x(0) = E$, the total energy in the pulse train. We then commented on the origin of the name, showing that the auto-correlation function is the cross-correlation function of a function with itself when that function is real.

In Section 33 the autocorrelation function went unmentioned until the very end where it was again rather trivially noted that $P = r_x(0)/T$ since P = E/T, and that $\mathcal{P}(\omega) = R_x(\omega)/(2\pi T)$ since $R_x(\omega) = |X(\omega)|^2$.

In Sections 34 and 35 no reference was made at all to the autocorrelation function.

This leads us to make several comments:

(1) The autocorrelation function played no role whatsoever in our development of key equations such as

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \left[(\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right] \qquad \text{// infinite} \qquad (35.7)$$

(2) In some textbooks, one gets the impression that the autocorrelation function is somehow crucial for the development of such equations. It is not.

(3) Nevertheless, since $\mathcal{P}(\omega) = R_x(\omega)/(2\pi T)$, one can start with a description of $x^{(i)}(t)$ of pulse train i in a statistical ensemble, compute from it $r_x^{(i)}(t)$ and from that $R_x^{(i)}(\omega)$. One could then do a statistical average to obtain $\langle \mathcal{P}(\omega) \rangle = (2\pi T)^{-1}(1/I) \sum_{i=1}^{I} R_x^{(i)}(\omega)$. So it is possible to take a pathway to deriving equations like (35.7) which does pass through the land of the autocorrelation function.

(4) Our main reason for even bringing it up is that the autocorrelation is closely related to what we are doing, and in other applications such as those involving noise (and pseudo-noise PN) it becomes more significant. In such applications, the definition of $r_x(t)$ might include a normalizing factor so it is then autocorrelation per unit time, or per pulse (per chip in the PN world).

(b) A paradox and its resolution

Now while we are here, we can back up and apply our statistical average directly to the spectrum in (36.2). Our result is

$$\langle X(\omega) \rangle = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \langle a_n \rangle\, e^{-in\omega T_1} \qquad (36.4)$$

We could then argue that

$$\langle a_n \rangle = [p]\, A + [1\text{-}p]\, B = p \qquad \text{if } A=1,\ B=0 \quad . \tag{36.5}$$

In this case, we can extract p from the sum in (36.4), which then collapses to form the usual (13.2) delta functions. The result is then the same as the regular pulse train result (14.4) with an overall factor of p out front, namely,

$$\langle X(\omega) \rangle = p \sum_{m=-\infty}^{\infty} X_{\mathbf{pulse}}(m\omega_1)\ 2\pi\ \delta(\omega T_1 - 2\pi m) \tag{36.6}$$

Thus says that our average spectrum is 100% *discrete*, there is no continuous part! If p = 1/2, the average spectrum is just 1/2 times our discrete unit amplitude pulse train spectrum (14.4). How can this be true, if we just showed in (35.10) that the average spectral density $\langle |X(\omega)|^2 \rangle$ has a *continuous* spectral component?

The answer lies in the fact that $\langle ab \rangle \neq \langle a \rangle\langle b \rangle$, where $\langle\ \rangle$ is our averaging operation. Thus

$$\langle |X(\omega)|^2 \rangle \neq \langle X(\omega) \rangle \langle X(\omega)^* \rangle \tag{36.7}$$

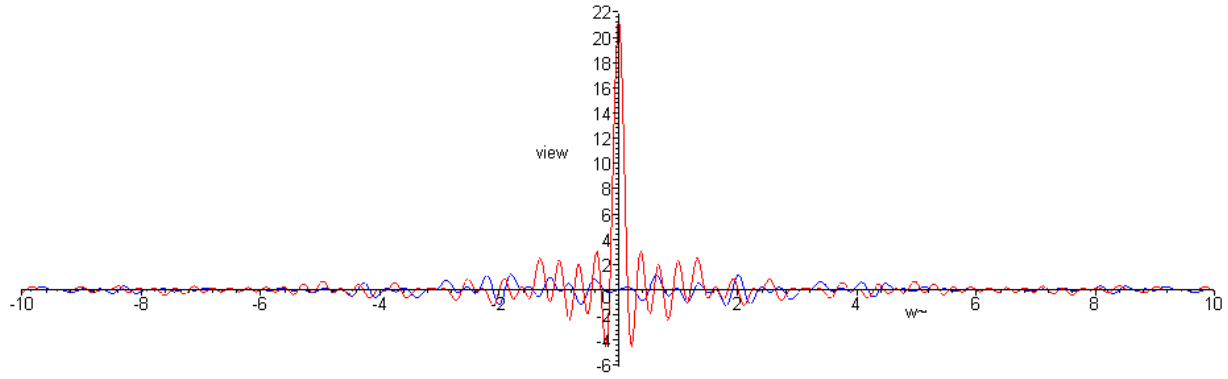
There is no reason in the world why the average of a product should be the product of the averages,

$$\left(\frac{1}{N} \Sigma_{i=1}^{N} A_i B_i \right) \neq \left(\frac{1}{N} \Sigma_{i=1}^{N} A_i \right) \left(\frac{1}{N} \Sigma_{i=1}^{N} B_i \right)$$

The HP Spectrum Analyzer measures $\langle |X(\omega)|^2 \rangle$, not $\langle X(\omega) \rangle$.

In the numerical example presented in Section 35 (e), we computed $\langle X(\omega) \rangle$ for a small ensemble of pulse trains. Here are plots of the real and imaginary part of $\langle X(\omega) \rangle$,

```
plot([Re(X_av),Im(X_av)], w=-10..10, view =-6..22 ,color = [red,blue],numpoints = 500);
```



We can see that, apart from the noise of our low statistics, there is only the central peak and no continuous spectrum component. A blow-up of the central region follows,

```
plot([Re(X_av),Im(X_av)], w=-4..4, view =-6..8 ,color = [red,blue],numpoints = 500);
```

