Question about the 1D string Green's function PhL 5.3.10

Stak always talks about a string on (0,ℓ), both in Chapter 1 and Chapter 4.

But I want to talk about a string on (-ℓ/2,ℓ/2). So make these replacements in his results:

x'= x - ℓ/2 so when his x = ℓ, my x' = ℓ/2.

So replace x = x' + ℓ/2 then rename x' to be x. Look then at his Green's function on page 261

$$g(x|\xi) = \Sigma_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)/\lambda_n = \Sigma_{n=1}^{\infty} (2/\ell) \sin(n\pi x/\ell) \sin(n\pi\xi/\ell) / [n\pi/\ell]^2$$

The lowest mode n = 1 has sin(nπx/ℓ) has a peak at x = ℓ/2 which will get moved now to 0, so it is the n = odd modes which peak in the center of the string. So I transcribe this into

$$g(x|\xi) = \Sigma_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)/\lambda_n = \Sigma_{n=1}^{\infty} (2/\ell) \sin(n\pi[x+\ell/2]/\ell) \sin(n\pi[\xi+\ell/2]/\ell) / [n\pi/\ell]^2$$

where $\varphi_n(x) = (2/\ell)^{1/2} \sin(n\pi[x+\ell/2]/\ell)$

Notice now that

$$\varphi_n(0) = (2/\ell)^{1/2} \sin(n\pi[\ell/2]/\ell) = (2/\ell)^{1/2} \sin(n\pi/2)$$

$= (2/\ell)^{1/2} (-1)^n$ n = odd // peaks mentioned above
$= 0$ n = even // no surprise here, central node

So we can say

$$g(x|0) = \Sigma_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)/\lambda_n = \Sigma_{n=odd} (2/\ell) (-1)^n \sin(n\pi[x+\ell/2]/\ell) / [n\pi/\ell]^2$$

$$= (\ell^2/\pi^2) (2/\ell) \Sigma_{n=odd} (-1)^n \sin(n\pi[x+\ell/2]/\ell)/n^2$$

I conjecture that

$$\sin(n\pi[x+\ell/2]/\ell) = \sin [n\pi x/\ell - n\pi/2] = (-1)^n \cos(n\pi x/\ell) \quad \text{for n odd}$$

so we would then have

$$g(x|0) =(\ell^2/\pi^2) (2/\ell) \Sigma_{n=odd} \cos(n\pi x/\ell)/n^2$$

$$= (2\ell/\pi^2) \Sigma_{n=odd} \cos(n\pi x/\ell)/n^2$$

Now consider this infinite series

$$S \equiv \Sigma_{n=1,3,5} \cos(ny)/n^2$$

Let n = 2m-1 so that m = 1,2,3.. then we have

$S \equiv \Sigma_{m=1,2,3} \cos([2m-1]y)/(2m-1)^2$

We voyage over to GR7 page 47 to get

$$6.^6 \qquad \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right) \qquad [-\pi \le x \le \pi] \qquad \text{FI III 546}$$

So we conclude that

$S = (\pi/4)(\pi/2 - |\alpha|)$

Therefore with $\alpha = \pi x/\ell$ we get

$g(x|0) = (2\ell/\pi^2) * (\pi/4)(\pi/2 - | \pi x/\ell |)$

$= (2\ell) * (1/4)(1/2 - | x/\ell |)$

$= (1/2)(\ell /2 - | x |)$

$= - (1/2)| x | + \ell/4$

This describes a down-facing V centered at x = 0 and having slope -1/2 on the right and +1/2 on the left. As we increase ℓ, the height of g in the center keeps increasing.

We can compare this to the 1D "fundamental solution" shown Stak p 51. This is normally taken to be simple E(x) = - (1/2)| x | where the constant is set to 0. My point is that we have obtained this result from the general form of the Green's function on the string $(-\ell/2,\ell/2)$ where

$g(x|\xi) = \Sigma_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)/\lambda_n = \Sigma_{n=1}^{\infty} (2/\ell) \sin(n\pi[x+\ell/2]/\ell) \sin(n\pi[\xi+\ell/2]/\ell) / [n\pi/\ell]^2$

$= \Sigma_{n=1}^{\infty} (2/\ell)(\ell/\pi)^2 \sin(n\pi[x+\ell/2]/\ell) \sin(n\pi[\xi+\ell/2]/\ell) /n^2$

$= (2\ell/\pi^2) \Sigma_{n=1}^{\infty} \sin(n\pi[x+\ell/2]/\ell) \sin(n\pi[\xi+\ell/2]/\ell) /n^2$

Now on scratch I have shown that

$\sin(n\pi[x+\ell/2]/\ell) = (-1)^n \sin(n\pi x/\ell)$ n even

$= (-1)^{n+1} \cos(n\pi x/\ell)$ n odd

so we can write our Green's function as

$$g(x|\xi) = (2\ell/\pi^2) \Sigma_{n=even} \sin(n\pi x/\ell) \sin(n\pi\xi/\ell) /n^2 - (2\ell/\pi^2) \Sigma_{n=odd} \cos(n\pi x/\ell) \cos(n\pi\xi/\ell) /n^2$$

I could go on to "do" all the series here to obtain Stak page 9 for the g, shifted by $\ell/2$,

$$g(x|\xi) = (1/\ell) (x_< + \ell/2)(\ell - [x_>+ \ell/2]) = - (1/\ell) (x_< + \ell/2) (x_< - \ell/2)$$

For $x < \xi$ this becomes

$$- (1/\ell) (x + \ell/2) (\xi - \ell/2) = - (x/\ell + 1/2) (\xi - \ell/2) = -x\xi/\ell + x/2 - \zeta/2 + \ell/4 = - (1/2)(\xi-x) -x\xi/\ell + \ell/4$$

and the result for $x > \xi$ is obtained by doing $x \leftrightarrow \xi$, which is like taking $\ell \rightarrow -\ell$ which you see is a symmetry of the form shown above. Thus, the result is

$$g(x|\xi) = - (1/2)|x-\xi| -x\xi/\ell + \ell/4$$

and as $\ell \rightarrow \infty$, this time we have a term $x\xi/\ell$ which vanishes, and we get the same $\ell/4$ constant which we ignore, and again we recover the fundy solution which is $g(x|\xi) =- (1/2)|x-\xi|$.

So you can ask: what happens to the Green's Function as $\ell \rightarrow \infty$. The answer is that the non-constant part stays fixed at $- (1/2)| x-\xi |$.

I conjecture that in 3D, something like this happens if we start with a sphere, do eigenfunctions, and take the sphere radius it to infinity -- the non-constant part will by $1/ |\mathbf{r}-\boldsymbol{\xi}|$.