These are just raw notes related to Stakgold page 40 top (vol 2). **7.8.09**

I had lots of trouble here, and cleaned it up in a later document called "parts and greens.doc".

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This entire section is garbage but I keep it anyway as an example of garbage:

A. Some examples of vLu-uL*v = ∇•J

Here I am examining for my first time the claim of Stak p 40 that vLu-uLv = **∇•J** . Trying to understand what would make this be true by looking at some simple examples. In general, want to show for an individual multiindex k term that:

$v\, a_{\mathbf{k}}(x) D^{\mathbf{k}} u - (-1)^{\mathbf{k}}\, u\, D^{\mathbf{k}}(a_{\mathbf{k}} v) = ? = \partial_{\mathbf{k}} J_{\mathbf{k}}$

In this little discussion, "level" = q means that all derivatives are assumed of order q.

1. Show for a simple first derivative with n = 1 and with a = 1

$v\ \partial u - (-1)^{\mathbf{1}}\ u\ \partial v = v\,\partial u + u\,\partial v = \partial(uv)$ = a divergence in 1D, so OK.

2. Show in case n = 2 with a = 1 and level 1:

$v\,[\partial_{\mathbf{x}} + \partial_{\mathbf{y}}]\ u - (-1) u\,[\partial_{\mathbf{x}} + \partial_{\mathbf{y}}]\, v = v\,[\partial_{\mathbf{x}} + \partial_{\mathbf{y}}]\, u + u\,[\partial_{\mathbf{x}} + \partial_{\mathbf{y}}]\, v$

$= (\partial_{\mathbf{x}} + \partial_{\mathbf{y}})(uv)$ J = (uv,uv) $\nabla \bullet J = \partial_{\mathbf{x}}(uv) + \partial_{\mathbf{y}}(uv)$

3. Show in case n = 2 with general a and level 1:

$v\,(a_{\mathbf{x}}\partial_{\mathbf{x}} + a_{\mathbf{y}}\partial_{\mathbf{y}})\, u + u\,\partial_{\mathbf{x}}[\, a_{\mathbf{x}} v] + \partial_{\mathbf{y}}[\, a_{\mathbf{y}} v]$

$= v\,(a_{\mathbf{x}}\partial_{\mathbf{x}} + a_{\mathbf{y}}\partial_{\mathbf{y}})\, u + u\,(a_{\mathbf{x}}\partial_{\mathbf{x}} + a_{\mathbf{y}}\partial_{\mathbf{y}})\, v + (uv)(\partial_{\mathbf{x}} a_{\mathbf{x}} + \partial_{\mathbf{y}} a_{\mathbf{y}})$

$= v\,(a_{\mathbf{x}}\partial_{\mathbf{x}}\, u + a_{\mathbf{y}}\partial_{\mathbf{y}}\, u) + u\,(a_{\mathbf{x}}\partial_{\mathbf{x}}\, v + a_{\mathbf{y}}\partial_{\mathbf{y}}\, v) + (uv)(\partial_{\mathbf{x}} a_{\mathbf{x}} + \partial_{\mathbf{y}} a_{\mathbf{y}})$

$= v\, \mathbf{a} \bullet \nabla u + u\, \mathbf{a} \bullet \nabla v + (uv)\, \nabla \bullet a$

$= \mathbf{a} \bullet \nabla(uv) + (uv)\, \nabla \bullet a$

$= \nabla \bullet [\, (uv)\, \mathbf{a}\,]$ QED

Do this again with summation notation

$v\,(a_i\partial_i u) + u\,\partial_i(a_i v) = v\,(a_i\partial_i u) + u\,(\partial_i a_i)v + u a_i(\partial_i v) = \partial_i(\,uva_i)$ QED

This proof applies to the case n = n with just level 1 derivatives.

4. Show in the case n = n with level 2 derivatives: [here I start running into problems...]

$L = a_k(x)D^k = a_{ij}\partial^{ij}$ implied sum on i and j with i + j = 2.

But what does ∂^{ij} mean? Below I show it as $\partial^i\partial^j$, but variables are not specified!!! Using this ill-defined notation, I here wander off into my "divergence approach" which is completely wrong.

B. Wandering down an erroneous road: the "divergence approach"

Then

$u\,a_{ij}\partial^{ij}v - v\,\partial^{ij}(a_{ij}u)$

$= u\,a_{ij}\partial^{ij}v - v\,\partial^i[\partial^j(a_{ij}u)]$

$= \partial^i\,\{\; - v\,[\partial^j(a_{ij}u)] + u\,[\partial^j(a_{ij}v)]\;\}$?? just a guess

$= - v\,\partial^i[\partial^j(a_{ij}u)] - (\partial^i v)\,[\partial^j(a_{ij}u)] + (\partial^i u)\,[\partial^j(a_{ij}v)] + u\,[\partial^{ji}(a_{ij}v)]$

What can we say about $\partial^{ij}(fg)$? Here I attempt to show that $\partial^{ij}(fg) = \partial^j J_j$ for some **J**, and then in the next little section I refer to **J** as **A**.

$\partial^{ij}(fg) = \partial^i[\partial^j(fg)\,] = \partial^i[f(\partial^j g) + g(\partial^j f)\,]$

$= f\,(\partial^{ij}g) + (\partial^i f)\,(\partial^j g) + (\partial^i g)\,(\partial^j f) + g\,(\partial^{ij}f)$

So using this rule, our result is then

$u\,a_{ij}\partial^{ij}v - v\,\partial^{ij}(a_{ij}u)$

$= u\,a_{ij}(\partial^{ij}v) - v\,\{\,a_{ij}\,(\partial^{ij}u) + (\partial^i a_{ij})\,(\partial^j u) + (\partial^j a_{ij})\,(\partial^i u) + (\partial^{ij}a_{ij})\,u\,\}$

$= a_{ij}\{\,u\,(\partial^{ij}v) - v\,(\partial^{ij}u)\,\} - (\partial^i a_{ij})\,v(\partial^j u) - (\partial^j a_{ij})\,v(\partial^i u) - (\partial^{ij}a_{ij})\,vu$

Only from below do I learn that this can be written in this form:

$-\partial^j J_j = \partial^j\,\{\; - v\,a_{ij}\,(\partial^i u) + u\,\partial^i(a_{ji}v)\,\}$ ≡ // I think

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End of my preliminary examples. At this point I am unaware of the general "parts integration" integral theorem and I am trying to find it. But I am focusing on divergences instead of gradients and this is taking me down the wrong path. Notation is still ill-defined.

The above seems awfully messy, as if it is the wrong approach. Let's go back to this claim

$$\int dV\ (vLu - uL^*v) = \int \mathbf{J}\bullet d\mathbf{S}$$

and consider our last case above which was this

$Lu = a_{ij}(\partial^{ij}u)$ where i+j = 2

$L^*v = \partial^{ij}\ (a_{ij}v)$ // Stak p 39 with + sign for all terms since level 2

Now let's consider the second integral on the LHS of our claimed theorem,

$$\int dV\ uL^*v\ =\ \int dV\ u\ \{\ \partial^{ij}\ (a_{ij}v)\ \}\ =\int dV\ u\ (\partial^{i}A_i)\ =\ \int dV\ u\ \nabla\bullet\mathbf{A}$$

where we define $A_i = \partial^j(a_{ij}v)$. Now I need a theorem like this:

$$\nabla\bullet\ (u\mathbf{A}) =\ \mathbf{A}\bullet\nabla u\ + u\nabla\bullet\mathbf{A}$$

So this tells us then that

$$\int dV\ uL^*v = \int dV\ u\ \nabla\bullet\mathbf{A}\ = \int dV\ \nabla\bullet(u\mathbf{A}) - \int dV\ \mathbf{A}\bullet\nabla u$$

Have we gotten anything useful by doing this? Can use divergence theorem on the first term to get

$$\int dV\ \nabla\bullet(u\mathbf{A}) = \int d\mathbf{S}\bullet(u\mathbf{A})$$

The second term is this

$$-\int dV\ \mathbf{A}\bullet\nabla u = -\int dV\ (\partial^{i}u)\ \partial^{j}(a_{ij}v)$$

In a sense, we have managed to move half the derivative off the v side and flop it over to the u side, so we have sort of accomplished the first of two "parts" integrations. I would now like to flop that ∂^j over to the u function somehow, maybe repeating the same idea. Suppose I define $B_j = a_{ij}v$ where i is a "parameter, so we can claim that $\partial^j(a_{ij}v) = \nabla\bullet\mathbf{B}$. Then our second term is this:

$$-\int dV\ \mathbf{A}\bullet\nabla u = -\int dV\ (\partial^i u)\ \partial^j(a_{ij}v) \ = \ -\int dV\ (\partial^i u)\ \nabla\bullet\mathbf{B}_{(i)}$$

where I show the parameter label. This then seems to have the same form as our u $\nabla\bullet\mathbf{A}$ integral, and we would then quote this theorem:

$$\nabla\bullet((\partial^i u)\,\mathbf{B}_{(i)}) = \mathbf{B}_{(i)}\bullet\nabla(\partial^i u) + (\partial^i u)\nabla\bullet\mathbf{B}_{(i)}$$

So our second term is now

$$-\int dV\ \mathbf{A}\bullet\nabla u = -\int dV\ (\partial^i u)\ \nabla\bullet\mathbf{B}_{(i)} = -\int dV\ \nabla\bullet((\partial^i u)\,\mathbf{B}_{(i)}) + \int dV\ \mathbf{B}_{(i)}\bullet\nabla(\partial^i u)$$

Again we can express the first term using the divergence theorem

$$-\int dV\ \nabla\bullet((\partial^i u)\,\mathbf{B}_{(i)}) = -\int d\mathbf{S}\bullet((\partial^i u)\,\mathbf{B}_{(i)})$$

The second term is then this:

$$\int dV\ \mathbf{B}_{(i)}\bullet\nabla(\partial^i u) = \int dV\ a_{ij}v\ \partial^j(\partial^i u) = \int dV\ v\ a_{ij}\,(\partial^{ij}u) = \int dV\ v\ (Lu)$$

Since this is a huge mess, we need now to retrace our steps:

$$\int dV\ u(L^*v) = \int dV\ u\ \nabla\bullet\mathbf{A} = \int dV\ \nabla\bullet(u\mathbf{A}) - \int dV\ \mathbf{A}\bullet\nabla u$$

$$= \int d\mathbf{S}\bullet(u\mathbf{A}) - \int dV\ \mathbf{A}\bullet\nabla u$$

$$= \int d\mathbf{S}\bullet(u\mathbf{A}) - \int dV\ (\partial^i u)\ \nabla\bullet\mathbf{B}_{(i)}$$

$$= \int d\mathbf{S}\bullet(u\mathbf{A}) - \int d\mathbf{S}\bullet((\partial^i u)\,\mathbf{B}_{(i)}) + \int dV\ \mathbf{B}_{(i)}\bullet\nabla(\partial^i u)$$

$$= \int d\mathbf{S}\bullet(u\mathbf{A}) - \int d\mathbf{S}\bullet((\partial^i u)\,\mathbf{B}_{(i)}) + \int dV\ v\ (Lu)$$

We then arrive at this conclusion:

$$\int dV\ \{\ v\ (Lu)\ -u(L^*v)\ \} = \int d\mathbf{S}\bullet\{\ ((\partial^i u)\,\mathbf{B}_{(i)}) - (u\mathbf{A})\ \}$$

and this looks like Stakgold 5.73 where $\mathbf{J} = ((\partial^i u)\,\mathbf{B}_{(i)}) - (u\mathbf{A})$. In components, we get

$$J_j = ((\partial^i u)\, B_{(i)j}) - (uA_j) = (\partial^i u)\, a_{ij} v - u\, \partial^i (a_{ji} v)$$

I think this example points the way to what a general proof might look like.

Here I try partitioning L into a sum of parts each of which has a "level" q.

Now let's try a more general term:

$$L = \Sigma_{|k| \le p}\ a_k D^k$$

and let's then partition this into "levels" so we have

$$L = \Sigma_{|k|=0}\ a_k D^k \ + \Sigma_{|k|=1}\ a_k D^k \ + \ldots + \ \Sigma_{|k|=p}\ a_k D^k$$

For example $\partial^2_x \partial_y$ is level 3 because 2 + 1 = 3. The level is the number of derivatives in each term. So in the level 3 we might have these terms

$$a_{21} \partial^2_1 \partial_2 + a_{30} \partial^3_1 + \text{etc.}$$

So now I will let q be the level index, and we have

$$L = \Sigma_{q=0}^{p} \left(\Sigma_{|k|=q}\ a_k D^k \right)$$

We are going to prove our theorem separately for each level q, so we need only deal with this simpler form of L which we could call L_q but I will just call it L

$$L = \Sigma_{|k|=q}\ a_k D^k \quad = \Sigma_{k1+k2+..+kn\ =q}\ a_{k1k2...kn}\, \partial_1^{k1} \partial_2^{k2}\, \partial_3^{k3} \ldots \partial_n^{kn}$$

I will now make this sum be "implicit" to avoid writing it all the time. Then our starting integral is going to be this:

$$\int dV\ (vLu - uL^*v)$$

$$= \int dV\ [\ (v\, a_{k1k2...kn}\, \partial_1^{k1} \partial_2^{k2}\, \partial_3^{k3} \ldots \partial_n^{kn}\, u$$

$$- u\, (-1)^q\, \partial_1^{k1} \partial_2^{k2}\, \partial_3^{k3} \ldots \partial_n^{kn}\, (a_{k1k2...kn}\, v)\ ]$$

The plan is to start with the second term and mess with it many times, and in the end we will pick up surface integral terms plus a final term that cancels the first term. In each step, we will move some derivatives from the "v" to the "u" until they all get moved. [this is the right plan, but I don't yet have the theorem that let's this plan work.]

Let's pick k_1 as our first "divergence" index. Then we write the second term in this way

$T2i = - u (-1)^q \partial_1^{k1} A_{k1} = \quad A_{k1} \equiv \partial_2^{k2} \partial_3^{k3} .... \partial_n^{kn} (a_{k1k2...kn} v)$

$= - (-1)^q [u \nabla \bullet \mathbf{A}]$

$- (-1)^q T2i = u \nabla \bullet \mathbf{A}$ // I guess OK to think of $\partial_1^{k1} A_{k1}$ as a divergence

The notation T2i means the Main Term 2 integrand. At each step we are going to apply this general theorem: [Ie, here is the "theorem" I am trying to use at this point.]

$\nabla\bullet (\varphi \mathbf{V}) = \mathbf{V} \bullet\nabla\varphi + \varphi\nabla\bullet\mathbf{V} \quad => \quad \varphi\nabla\bullet\mathbf{V} = \nabla\bullet (\varphi\mathbf{V}) - \mathbf{V} \bullet\nabla\varphi$

In our first application, we have $\varphi = u$ and $\mathbf{V} = \mathbf{A}$. We then have

$- (-1)^q T2i = u \nabla \bullet \mathbf{A} = \nabla\bullet (u \mathbf{A}) - \nabla u \bullet \mathbf{A} = \nabla\bullet (u \mathbf{A}) - (\partial^{k1} u) A_{k1}$

$= \nabla\bullet (u \mathbf{A}) - (\partial^{k1} u) [\partial_2^{k2} \partial_3^{k3} .... \partial_n^{kn} (a_{k1k2...kn} v)]$

Here at once we have our first "divergence term" and the residual term has "one derivative set" moved over from v to u. Whatever derivatives there were wrt variable x_1, then are

STOP! Everything just fell apart, my prototype case was done wrong as well.

In the above I was wrongly thinking of $\partial_1^k A_k$ as a "divergence", but of course $\partial_k A_k$ is the actual divergence of A. My notation was hopelessly confused, almost everything above is wrong. I was not clear about which index was "the variable" and which index was "the order of the derivative". Very bad.

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C. Phil discovers and proves by induction the correct parts integration theorem.

Return to prototype case.

$\int dV (vLu - uL^*v) = \int \mathbf{J}\bullet d\mathbf{S}$

and consider our last case above which was this

$Lu = a_{ij}(\partial^{ij} u)$ where i+j = 2

$L^*v = \partial^{ij} (a_{ij} v)$ // Stak p 39 with + sign for all terms since level 2

Here I finally realize my notation was ill defined.

The notation ∂^{ij} really means "i" derivatives wrt x_1 and also "j" derivatives wrt x_2. Write it out

$$\partial^{ij} = \partial_1{}^i \partial_2{}^j$$

Now let's consider the second integral on the LHS of our claimed theorem,

$$\int dV\ uL^*v = \int dV\ u \{ \partial^{ij} (a_{ij} v) \} = \int dV\ u \{ \partial_1{}^i \partial_2{}^j (a_{ij} v) \}$$

Unfortunately, I don't see any "divergences" here. I think our theorem has to be true for each particular set of {i,j} values, so treat them as constants, not summation indices! [So my entire "divergence" approach in the last section is useless, besides being completely erroneous.]

So, what can we do with this integral??? Maybe I should try the case i = j = 1 to find a pathway.

$$\int d^3x\ u(x,y) \{ \partial_x \partial_y (a_{11}(x,y) v(x,y)) \}$$

Here I finally stumble onto the meaning of parts integration in R_n.

I don't know how to do "parts" with an individual variable like x. Well, I have this theorem:
[this is the "volume integral of a gradient" theorem, I don't know any other name for it.]

$$\int dV\ \nabla\varphi = \int d\mathbf{S}\ \varphi$$

Suppose I dot both sides of this equation with a fixed unit vector $\hat{\mathbf{i}}$ to get

$$\int dV\ \hat{\mathbf{i}} \bullet \nabla\varphi = \int dV\ (\partial_i \varphi) = \int d\mathbf{S} \bullet \hat{\mathbf{i}}\ \varphi = \int dS_i\ \varphi$$

This gives the following instruction: for each surface patch d**S** on the surface integral, consider only the patch component in the i direction and use that in the integration on the right. I guess that is OK. Then we will end up with this "parts" formula

$$\int_V dV\ \psi(\partial_i \varphi) = -\int_V dV\ (\partial_i \psi)\varphi + \int_S dS \bullet \hat{\mathbf{i}}\ (\psi\varphi)$$

So I guess I really do know how to do a single variable parts integration in R_n. Now make this replacement in the above

$$\varphi \rightarrow \partial_i \varphi$$

Then our parts theorem says

$$\int_V dV\ \psi(\partial_i^2\varphi) = -\int_V dV\ (\partial_i\psi)\,(\partial_i\varphi) + \int_S dS\bullet\hat{\mathbf{i}}\ [\psi(\partial_i\varphi)]$$

Now apply parts a second time on the $(\partial_i\psi)\,(\partial_i\varphi)$ term and it becomes

$$\int_V dV\ (\partial_i\psi)\,(\partial_i\varphi) = -\int_V dV\ (\partial_i(\partial_i\psi))\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ ((\partial_i\psi)\varphi)$$

So gather things up and we have shown that

$$\int_V dV\ \psi(\partial_i^2\varphi) = \int_V dV\ (\partial_i^2\psi)\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ [\psi(\partial_i\varphi)] - \int_S dS\bullet\hat{\mathbf{i}}\ ((\partial_i\psi)\varphi)$$

$$= \int_V dV\ (\partial_i^2\psi)\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ \{\ \psi(\partial_i\varphi) - (\partial_i\psi)\varphi\ \}$$

Let's now go another step. Replace again $\varphi \rightarrow \partial_i\varphi$ to get

$$\int_V dV\ \psi(\partial_i^3\varphi) = \int_V dV\ (\partial_i^2\psi)\,(\partial_i\varphi) + \int_S dS\bullet\hat{\mathbf{i}}\ \{\ \psi(\partial_i^2\varphi) - (\partial_i\psi)\,(\partial_i\varphi)\ \}$$

Now do parts on the first term on the right:

$$\int_V dV\ (\partial_i^2\psi)\,(\partial_i\varphi) = -\int_V dV\ (\partial_i^3\psi)\,\varphi + \int dS\bullet\hat{\mathbf{i}}\ (\partial_i^2\psi)\ \varphi$$

We then arrive at the following result

$$\int_V dV\ \psi(\partial_i^3\varphi) = -\int_V dV\ (\partial_i^3\psi)\,\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ \{\ \psi(\partial_i^2\varphi) - (\partial_i\psi)\,(\partial_i\varphi) + (\partial_i^2\psi)\,\varphi\ \}$$

I have done something like this all before recently, but not quite. Here is a **conjectured** general case:

$$\int_V dV\ \psi(\partial_i^k\varphi) = (-1)^k \int_V dV\ (\partial_i^k\psi)\,\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ \{\ \Sigma_{m=0}^{k-1}\,(-1)^m\,(\partial_i^m\psi)\,(\partial_i^{k-m-1}\varphi)\ \}$$

For the case k = 1 this says:

$$\int_V dV\ \psi(\partial_i\varphi) = (-1)^k \int_V dV\ (\partial_i\psi)\,\varphi + \int_S dS\bullet\hat{\mathbf{i}}\ \{\ (\psi)\,(\varphi)\ \}$$

which is our usual result. Now do an **induction proof**. Assume true for k. Then show true for k+1.

$$\int_V dV\ \psi(\partial_i^{k+1}\varphi) = \int_V dV\ \psi\ \partial_i\,(\partial_i^k\varphi)$$

$$= -\int_V dV(\partial_i\psi)\,(\partial_i{}^k\varphi) + \int dS\bullet\hat{\mathbf{i}}\;\psi\,(\partial_i{}^k\varphi) \qquad \text{// from our single parts formula}$$

Now use the level k formula for the first term using $\psi \rightarrow (\partial_i\psi)$ to get

$$\int_V dV\,\psi(\partial_i{}^k\varphi) = (-1)^k \int_V dV\,(\partial_i{}^k\psi)\,\varphi + \int_S dS\bullet\hat{\mathbf{i}}\;\{\Sigma_{n=0}{}^{k-1}(-1)^n(\partial_i{}^n\psi)(\partial_i{}^{k-n-1}\varphi)\}$$

$$\int_V dV\,(\partial_i\psi)(\partial_i{}^k\varphi) = (-1)^k \int_V dV\,(\partial_i{}^{k+1}\psi)\,\varphi + \int_S dS\bullet\hat{\mathbf{i}}\;\{\Sigma_{n=0}{}^{k-1}(-1)^n(\partial_i{}^{n+1}\psi)(\partial_i{}^{k-n-1}\varphi)\}$$

Then we have shown that

$$\int_V dV\,\psi(\partial_i{}^{k+1}\varphi) = -(-1)^k \int_V dV\,(\partial_i{}^{k+1}\psi)\,\varphi - \int_S dS\bullet\hat{\mathbf{i}}\;\{\Sigma_{n=0}{}^{k-1}(-1)^n(\partial_i{}^{n+1}\psi)(\partial_i{}^{k-n-1}\varphi)\}$$

$$+ \int dS\bullet\hat{\mathbf{i}}\;\psi\,(\partial_i{}^k\varphi)$$

$$= (-1)^{k+1}\int_V dV\,(\partial_i{}^{k+1}\psi)\,\varphi$$

$$+ \int dS\bullet\hat{\mathbf{i}}\;[\psi\,(\partial_i{}^k\varphi) - \{\Sigma_{n=0}{}^{k-1}(-1)^n(\partial_i{}^{n+1}\psi)(\partial_i{}^{k-n-1}\varphi)\}]$$

Now define a new summation index n' = n+1 so [..] becomes

$$[\psi\,(\partial_i{}^k\varphi) - \{\Sigma_{n'=1}{}^{k}(-1)^{n'-1}(\partial_i{}^{n'}\psi)(\partial_i{}^{k-n'}\varphi)\}]$$

$$= [\psi\,(\partial_i{}^k\varphi) + \{\Sigma_{n'=1}{}^{k}(-1)^{n'}(\partial_i{}^{n'}\psi)(\partial_i{}^{k-n'}\varphi)\}]$$

$$= [\psi\,(\partial_i{}^k\varphi) + \{\Sigma_{n=1}{}^{k}(-1)^{n}(\partial_i{}^{n}\psi)(\partial_i{}^{k-n}\varphi)\}]$$

But the first term is the n=0 term in the series, so get

$$= [\Sigma_{n=0}{}^{k}(-1)^n(\partial_i{}^n\psi)(\partial_i{}^{k-n}\varphi)]$$

$$= [\Sigma_{n=0}{}^{(k+1)-1}(-1)^n(\partial_i{}^n\psi)(\partial_i{}^{(k+1)-n-1}\varphi)]$$

And we have now shown that

$$\int_V dV\,\psi(\partial_i{}^{k+1}\varphi) = (-1)^{k+1}\int_V dV\,(\partial_i{}^{k+1}\psi)\,\varphi$$

$$+ \int dS\bullet\hat{\mathbf{i}}\;[\Sigma_{n=0}{}^{(k+1)-1}(-1)^n(\partial_i{}^n\psi)(\partial_i{}^{(k+1)-n-1}\varphi)]$$

But this is exactly the k+1 result obtained from the conjectured form for k, so this concludes our induction proof. We now have this very important result I never knew about:

Theorem: You can do parts k times wrt a particular coordinate "i" derivative in R_n to get:

$$\int_V dV\, \psi(\partial_i{}^k\varphi) = (-1)^k \int_V dV\, (\partial_i{}^k\psi)\, \varphi + \int_S d\mathbf{S}\bullet\hat{\mathbf{i}}\ \{ \Sigma_{m=0}{}^{k-1} (-1)^m (\partial_i{}^m\psi)(\partial_i{}^{k-m-1}\varphi) \}$$

$$= (-1)^k \int_V dV\, (\partial_i{}^k\psi)\, \varphi + \int_S d\mathbf{S} \bullet \mathbf{J}^{(k)}$$

where $\mathbf{J}^{(k)} = \hat{\mathbf{i}}\ \{ \Sigma_{m=0}{}^{k-1} (-1)^m (\partial_i{}^m\psi)(\partial_i{}^{k-m-1}\varphi) \}$. But we can then use the divergence theorem on the last term to write it as:

$$\int_S d\mathbf{S} \bullet \mathbf{J}^{(k)} = \int_V dV\, \nabla \bullet \mathbf{J}^{(k)}$$

Comparing things, we have shown that

$$\psi(\partial_i{}^k\varphi) - (-1)^k(\partial_i{}^k\psi)\, \varphi = \nabla \bullet \mathbf{J}^{(k)} = \partial_i \{ \Sigma_{n=0}{}^{k-1} (-1)^n (\partial_i{}^n\psi)(\partial_i{}^{k-n-1}\varphi) \}$$

Now we are well on our way to proving the general result! This is all new stuff for me.

D. Proof of Stakgold Claims on page 40 Top

First, here is our general L operator

$$L = \Sigma_{|k|\le p}\ a_k D^k$$

$$= \Sigma_{|k|\le p}\ a_{k1k2\ldots kn}\, \partial_1{}^{k1}\partial_2{}^{k2}\, \partial_3{}^{k3} \ldots \partial_n{}^{kn}$$

We don't have to partition this in any way, we just take a particular term and show that our theorem applies to each term. So simplify and say

$$L = a_k D^k = a_{k1k2\ldots kn}\, \partial_1{}^{k1}\partial_2{}^{k2}\, \partial_3{}^{k3} \ldots \partial_n{}^{kn}$$

The LHS of our theorem is then this (for our particular term)

$$\int dV\, (vLu - uL^*v) = \int dV\, \{ v\, a_k D^k u - (-1)^k u\, D^k(a_k v) \}$$

$$\int dV\, [\ (v\, a_{k1k2\ldots kn}\, \partial_1{}^{k1}\partial_2{}^{k2}\, \partial_3{}^{k3} \ldots \partial_n{}^{kn}\, u$$
$$- u\, (-1)^{k1+k2\ldots+kn}\, \partial_1{}^{k1}\partial_2{}^{k2}\, \partial_3{}^{k3} \ldots \partial_n{}^{kn}\, (a_{k1k2\ldots kn}\, v)\ ] \quad = T1 - (-1)^{k1+k2\ldots+kn}\, T2$$

$T2 = \int dV\ u\ [\partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$

Our first action is to move the ∂_1^{k1} in T2 from the right to the left. First, we state our theorem from above, first setting $k = k_1$ and $i = 1$.

$$\int_V dV\ \psi(\partial_1^{k1}\varphi) = (-1)^{k1} \int_V dV\ (\partial_1^{k1}\psi)\ \varphi + \int_S d\mathbf{S}\bullet\hat{\mathbf{1}}\ \{\ \Sigma_{m=0}^{k1-1}\ (-1)^m\ (\partial_1^m\psi)\ (\partial_1^{k1-m-1}\varphi)\ \}$$

Now set $\psi = u$ and $\varphi = \partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)$. Then we get for T2

$$T2 = \int_V dV\ u\ [\partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$$

$$= (-1)^{k1} \int_V dV\ (\partial_1^{k1}\ u)\ [\partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$$

$$+ \int_S d\mathbf{S}\bullet\hat{\mathbf{1}}\ \{\ \Sigma_{m=0}^{k1-1}\ (-1)^m\ (\partial_1^m\ u)\ \ \partial_1^{k1-m-1}\ [\partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]\ \}$$

$$= (-1)^{k1}\ T21 + T22$$

where

$$T21 = \int_V dV\ (\partial_1^{k1}\ u)\ [\partial_2^{k2}\ \partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$$

Now apply our theorem again to T21. Again quote the theorem but now with $i = 2$ and $k = k2$:

$$\int_V dV\ \psi(\partial_2^{k2}\varphi) = (-1)^{k2} \int_V dV\ (\partial_2^{k2}\psi)\ \varphi + \int_S d\mathbf{S}\bullet\hat{\mathbf{2}}\ \{\ \Sigma_{m=0}^{k2-1}\ (-1)^m\ (\partial_2^m\psi)\ (\partial_2^{k2-m-1}\varphi)\ \}$$

Then set $\psi = (\partial_1^{k1}\ u)$ and $\varphi = [\partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$ to get

$$T21 = \int_V dV\ (\partial_1^{k1}\ u)\ (\partial_2^{k2}[\partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ])$$

$$= (-1)^{k2} \int_V dV\ (\partial_2^{k2}(\partial_1^{k1}\ u))\ [\partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ]$$

$$+ \int_S d\mathbf{S}\bullet\hat{\mathbf{2}}\ \{\ \Sigma_{m=0}^{k2-1}\ (-1)^m\ (\partial_2^m(\partial_1^{k1}\ u))\ (\partial_2^{k2-m-1}\ [\partial_3^{k3}\ ....\ \partial_n^{kn}\ (a_{k1k2...kn}\ v)\ ])\ \}$$

$$= (-1)^{k2}\ T211 + T212$$

Next, we apply our theorem to integral T211

$$T211 = \int_V dV\ (\partial_2^{k2}\partial_1^{k1}\ u))\ [\partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ]$$

Quote first with i = 3 and k = k3:

$$\int_V dV\ \psi(\partial_3^{k3}\varphi) = (-1)^{k3} \int_V dV\ (\partial_3^{k3}\psi)\ \varphi + \int_S dS\bullet\hat{3}\ \{ \Sigma_{m=0}^{k3-1}\ (-1)^m\ (\partial_3^m\psi)\ (\partial_3^{k3-m-1}\varphi)\ \}$$

Then set $\psi = (\partial_2^{k2}\partial_1^{k1}\ u)$ and $\varphi = [\partial_4^{k4} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ]$.

$$T211 = \int_V dV\ (\partial_2^{k2}\partial_1^{k1}\ u)\ (\partial_3^{k3}\ [\partial_4^{k4} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ])$$

$$= (-1)^{k3} \int_V dV\ (\partial_3^{k3}(\partial_2^{k2}\partial_1^{k1}\ u)\ [\partial_4^{k4} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ]$$

$$+ \int_S dS\bullet\hat{3}\ \{ \Sigma_{n=m}^{k3-1}\ (-1)^m\ (\partial_3^m(\partial_2^{k2}\partial_1^{k1}\ u)\ (\partial_3^{k3-m-1}\ [\partial_4^{k4} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ])\ \}$$

At this point we have this total result:

$$\int dV\ [\ (v\ a_{k1k2\ldots kn}\ \partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3} \ldots \partial_n^{kn}\ u$$
$$- u\ (-1)^{k1+k2\ldots+kn}\ \partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ] \quad = T1 - (-1)^{k1+k2\ldots+kn}\ T2$$

$$T2 = \int dV\ u\ [\partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ] \quad = (-1)^{k1}\ T21 + T22$$

$$T21 = \int_V dV\ (\partial_1^{k1}\ u)\ [\partial_2^{k2}\ \partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ] = (-1)^{k2}\ T211 + T212$$

$$T211 = \int_V dV\ (\partial_2^{k2}\partial_1^{k1}\ u))\ [\partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ] = (-1)^{k3}\ T2111 + T2112$$

When the dust settles, we can see that we will pick up an overall sign $(-1)^{k1+k2\ldots+kn}$ as the first term propagates to the end of the line, and this will cancel the same sign factor in $T1 - (-1)^{k1+k2\ldots+kn}\ T2$. We will then have, in terms of this first term propagation, a duplicate of the first term of our theorem but with the minus sign sitting in front of the u above, so these terms will cancel. We end up with nothing but the surface terms as our final result. Here are the surface terms we picked up above, with sign factors:

(1) $- (-1)^{k1+k2\ldots+kn}\ T22$

$$= - (-1)^{k1+k2\ldots+kn} \int_S dS\bullet\hat{1}\ \{ \Sigma_{m=0}^{k1-1}\ (-1)^m\ (\partial_1^m\ u)\ \partial_1^{k1-m-1}\ [\partial_2^{k2}\ \partial_3^{k3} \ldots \partial_n^{kn}\ (a_{k1k2\ldots kn}\ v)\ ]\ \}$$

(2) $- (-1)^{k1+k2\ldots+kn}\ (-1)^{k1}\ T212$

$$= -(-1)^{k2\ldots+kn} \int_S d\mathbf{S}\bullet\hat{\mathbf{2}} \ \{ \Sigma_{m=0}^{k2-1} (-1)^m (\partial_2^m(\partial_1^{k1} u)) (\partial_2^{k2-m-1} [\partial_3^{k3} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)]) \ \}$$

$$(3) -(-1)^{k2\ldots+kn} \ (-1)^{k2} \ T2112$$

$$= -(-1)^{k3\ldots+kn} \int_S d\mathbf{S}\bullet\hat{\mathbf{3}} \ \{ \Sigma_{m=0}^{k3-1} (-1)^m (\partial_3^m(\partial_2^{k2}\partial_1^{k1} u) (\partial_3^{k3-m-1} [\partial_4^{k4} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)]) \ \}$$

I will now write the currents for each of these terms

$$J_1 = -(-1)^{k1+k2\ldots+kn} \Sigma_{m=0}^{k1-1} (-1)^m (\partial_1^m u) \ \partial_1^{k1-m-1} [\partial_2^{k2} \partial_3^{k3} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)]$$

$$J_2 = -(-1)^{k2\ldots+kn} \quad \Sigma_{m=0}^{k2-1} (-1)^m (\partial_2^m\partial_1^{k1} u)) (\partial_2^{k2-m-1} [\partial_3^{k3} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)])$$

$$J_3 = -(-1)^{k3\ldots+kn} \quad \Sigma_{m=0}^{k3-1} (-1)^m (\partial_3^m\partial_2^{k2}\partial_1^{k1} u) (\partial_3^{k3-m-1} [\partial_4^{k4} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)])$$

The general formula appears to be this:

$$J_s = -(-1)^{ks\ldots+kn} \Sigma_{m=0}^{ks-1} (-1)^m$$
$$(\partial_s^m\partial_{s-1}^{k(s-1)}\ldots\partial_2^{k2}\partial_1^{k1} u) (\partial_s^{ks-m-1} [\partial_{s+1}^{k(s+1)} .... \partial_n^{kn} (a_{k1k2\ldots kn} v)])$$

The last current will then be

$$J_n = -(-1)^{kn} \Sigma_{m=0}^{kn-1} (-1)^m$$
$$(\partial_n^m\partial_{n-1}^{k(n-1)}\ldots\partial_2^{k2}\partial_1^{k1} u) (\partial_n^{kn-m-1} [(a_{k1k2\ldots kn} v)])$$

As you can see, looking at the general current component J_s, the current is bilinear in u and v (allowing for derivatives) and is linear in the a_k coefficient (again allowing for derivatives).

The upshot of all this work is that we have shown this fact for a particular term "k" in operator L:

$$\int dV \ (vLu - uL^*v) \ = \ \int dV \ \{ \ v \ a_k D^k u \ - (-1)^k \ v \ D^k(a_k u) \ \} \ = \int_S d\mathbf{S} \bullet \mathbf{J}$$

where the vector current **J** is as shown above. We can then apply the divergence theorem to write this last term as $\int dV \ \nabla\bullet J$ and then arrive at this fact,

$$vLu - uL^*v \ = \nabla\bullet\mathbf{J}$$

Since this is true for each term in L, it is true for L as the sum of all its terms. We would then add a general multiindex k sum to the current components shown above. Thus we have proven the claim that Stakgold makes on page 40 top.

Example 1: Let $L = \nabla^2$ so that $a_k = 1$. The first term ∂_1^2 has k1 = 2 and all other $k_i = 0$. This means that all the current components above except J_1 vanish (for this term) and we get

$$J_1 = -(-1)^2 \Sigma_{m=0}^{1} (-1)^m (\partial_1^m u)\ \partial_1^{1-m} v$$

$$= -\{ u\, \partial_1 v\ - (\partial_1 u)\, v \}$$

Let's now explicitly compute the ∂_2^2 term since we are having problems with Example 2 below. This term has k2 = 2 and other $k_i = 0$. so get

$$J_2 = -(-1)^2 \Sigma_{m=0}^{1} (-1)^m (\partial_2^m u))\ (\partial_2^{1-m} [(v)\ ])$$

So this term really does look just like the first term with 1→2.

If we look at the ∂_2^2 term we get a similar result, and I conclude that

$$\mathbf{J} = -\{ u\, \nabla v\ - (\nabla u)\, v \} \quad = v\nabla u - u\nabla v \qquad \text{// agrees with 5.74}$$

The following Example 2 consumes the entire rest of this document! See "parts and greens.." for an explanation of what is going on here. The moral is that the current J is not unique.

First do it the painful Phil way:

Example 2: (Painful!) Let $L = \nabla^4$ so again $a_k = 1$. The terms are these

$$L = (\Sigma_i \partial_i^2)(\Sigma_j \partial_j^2) = \Sigma_{ij=1}^{n}\ \partial_i^2 \partial_j^2$$

If we consider just the term $\partial_i^2 \partial_j^2$ with $i \neq j$ we have $k_i = k_j = 2$ and all other $k_s = 0$. For such a term, we will only have current components J_i and J_j since the m sums are 0 in all other components. Then if i > j

$$J_s = -(-1)^{ks\ldots+kn} \Sigma_{m=0}^{ks-1} (-1)^m$$
$$(\partial_s^m \quad [\partial_{s-1}^{k(s-1)} \ldots \partial_2^{k2} \partial_1^{k1} u\ ])\quad (\partial_s^{ks-m-1} [\partial_{s+1}^{k(s+1)} \ldots \partial_n^{kn} (a_{k1k2\ldots kn} v)\ ])$$
up down up

$$J_i = -(-1)^2 \Sigma_{m=0}^{1} (-1)^m\ \ (\partial_i^m\ \ [\ \partial_j^2 u\ ]\ )\ (\partial_i^{1-m} [\ \ (v)\ \ ]) \qquad i > j$$

$$= -\ \{(\partial_j^2 u)\ (\partial_i v)\ -\ (\partial_i \partial_j^2 u)\ v\ \}$$

$$J_j = \ -(-1)^{2+2} \Sigma_{m=0}^{1} (-1)^m\ (\ \partial_j^m u)(\partial_j^{1-m}\, \partial_i^2 v) \qquad i > j$$

$$= -\{ u\ (\partial_j \partial_i^2 v)\ -(\partial_j u)\ (\partial_i^2 v\ )$$

Question Added: What are the above results in the case that i < j ? Let's do this by brute force:
up

$$J_s = -(-1)^{ks\ldots+kn} \sum_{m=0}^{ks-1} (-1)^m$$
$$(\partial_s^m \quad [\partial_{s-1}^{k(s-1)} \ldots \partial_2^{k2} \partial_1^{k1} u\,]) \quad (\partial_s^{ks-m-1} [\partial_{s+1}^{k(s+1)} \ldots \partial_n^{kn} (a_{k1k2\ldots kn} v)\,])$$
down up

$$J_i = -(-1)^{2+2} \sum_{m=0}^{1} (-1)^m \quad (\partial_i^m \quad [u\,])\,(\partial_i^{1-m} [\partial_j^2 (v)\,]) \qquad i < j$$

$$= - \{(u)\,(\partial_i \partial_j^2 v) \;-\; (\partial_i u)\,(\partial_j^2 v)\,\} \qquad ***$$

$$J_j = -(-1)^2 \sum_{m=0}^{1} (-1)^m \,(\,\partial_j^m \,\partial_i^2 u)(\partial_j^{1-m} v) \qquad i < j$$

$$= - \{\,(\,\partial_i^2 u)(\partial_j v) \;-(\partial_j \,\partial_i^2 u)\,(v\,) \qquad ****$$

As expected, these results are what you get from the first results making $i \leftrightarrow j$ everywhere.

Now if i = j, we get only one term which will have $k_i = 4$ and all others 0, so

$$J_i = -(-1)^4 \sum_{m=0}^{3} (-1)^m \;(\partial_i^m u)\,(\partial_i^{3-m} v)$$

$$= - \{\, u\,(\partial_i^3 v) - (\partial_i u)\,(\partial_i^2 v) + (\partial_i^2 u)\,(\partial_i v) - (\partial_i^3 u)\, v\,\}$$

So let's write our sum like this

$$L = \sum_{i=1}^{n} \partial_i^4 \;+ 2 \sum_{i>j} \partial_i^2 \partial_j^2$$

The total current is then:

$$-\mathbf{J} = \sum_{i=1}^{n} \;[\;\hat{\mathbf{i}}\,\{\, u\,(\partial_i^3 v) - (\partial_i u)\,(\partial_i^2 v) + (\partial_i^2 u)\,(\partial_i v) - (\partial_i^3 u)\, v\,\}\,]$$

$$+ 2 \sum_{i>j} \;\hat{\mathbf{i}}\,[\,(\partial_j^2 u)\,(\partial_i v) \;-\; (\partial_i \partial_j^2 u)\, v\,]$$

$$+ 2 \sum_{i>j} \;\hat{\mathbf{j}}\,[\, u\,(\partial_j \partial_i^2 v) \;-(\partial_j u)\,(\partial_i^2 v\,)\,]$$

In the last term, i and j really are dummy summation indices, so I should be able to relabel them:

$$-\mathbf{J} = \sum_{i=1}^{n} \;[\;\hat{\mathbf{i}}\,\{\, u\,(\partial_i^3 v) - (\partial_i u)\,(\partial_i^2 v) + (\partial_i^2 u)\,(\partial_i v) - (\partial_i^3 u)\, v\,\}\,]$$

$$+ 2 \sum_{i>j} \;\hat{\mathbf{i}}\,[\,(\partial_j^2 u)\,(\partial_i v) \;-\; (\partial_i \partial_j^2 u)\, v\,]$$

$$+ 2 \sum_{j>i} \;\hat{\mathbf{i}}\,[\, u\,(\partial_i \partial_j^2 v) \;-(\partial_i u)\,(\partial_j^2 v\,)\,]$$

Comment: Can we check at this point whether $\nabla\bullet\mathbf{J}$ has the proper u v symmetry?

$$\nabla\bullet(-\mathbf{J}) \;=\; \partial_k (-J)_k$$

But from the above we have that

$$(-J)_k = -\mathbf{J} \bullet \hat{\mathbf{k}} = \Sigma_{i=1}^{n} \left[\delta_{ik} \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \} \right]$$

$$+ 2 \Sigma_{i>j} \delta_{ik} \left[(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v \right]$$

$$+ 2 \Sigma_{j>i} \delta_{ik} \left[u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v) \right]$$

$$= \{ u (\partial_k^3 v) - (\partial_k u)(\partial_k^2 v) + (\partial_k^2 u)(\partial_k v) - (\partial_k^3 u) v \}$$

$$+ 2 \theta(k-j) \left[(\partial_j^2 u)(\partial_k v) - (\partial_k \partial_j^2 u) v \right]$$

$$+ 2 \theta(j-k) \left[u (\partial_k \partial_j^2 v) - (\partial_k u)(\partial_j^2 v) \right]$$

So the divergence is given by

$$\nabla \bullet (-\mathbf{J}) = \partial_k (-J)_k = \partial_k \{ u (\partial_k^3 v) - (\partial_k u)(\partial_k^2 v) + (\partial_k^2 u)(\partial_k v) - (\partial_k^3 u) v \}$$

$$+ 2 \theta(k-j) \partial_k \left[(\partial_j^2 u)(\partial_k v) - (\partial_k \partial_j^2 u) v \right]$$

$$+ 2 \theta(j-k) \partial_k \left[u (\partial_k \partial_j^2 v) - (\partial_k u)(\partial_j^2 v) \right]$$

$$= \partial_k \{ u (\partial_k^3 v) - (\partial_k u)(\partial_k^2 v) + (\partial_k^2 u)(\partial_k v) - (\partial_k^3 u) v \}$$

$$+ 2 \Sigma_{k>j} \partial_k \left[(\partial_j^2 u)(\partial_k v) - (\partial_k \partial_j^2 u) v \right]$$

$$+ 2 \Sigma_{j>k} \partial_k \left[u (\partial_k \partial_j^2 v) - (\partial_k u)(\partial_j^2 v) \right]$$

The first term has our required symmetry, so it is the last two terms that are of concern. Let's change the dummy indices in the last line:

$$+ 2 \Sigma_{k>j} \partial_k \left[(\partial_j^2 u)(\partial_k v) - (\partial_k \partial_j^2 u) v \right]$$

$$+ 2 \Sigma_{k>j} \partial_j \left[u (\partial_j \partial_k^2 v) - (\partial_j u)(\partial_k^2 v) \right]$$

$$= 2 \Sigma_{k>j} \{ (\partial_k \partial_j^2 u)(\partial_k v) + (\partial_j^2 u)(\partial_k^2 v) - (\partial_k^2 \partial_j^2 u) v - (\partial_k \partial_j^2 u)(\partial_k v)$$

$$+ (\partial_j u)(\partial_j \partial_k^2 v) + u(\partial_j^2 \partial_k^2 v) - (\partial_j^2 u)(\partial_k^2 v) - (\partial_j u)(\partial_j \partial_k^2 v) \}$$

$$= 2 \Sigma_{k>j} \{ - v (\partial_k^2 \partial_j^2 u) + u(\partial_j^2 \partial_k^2 v) \}$$

and after cancellation of terms, we do indeed have our desired symmetry!

Plan A for debug (earlier). Suppose we had just $L = \partial_1^2\partial_2^2$. This L is formally self adjoint, so I would expect to have J(v,u) = - J(u,v) . Let's look at our results from above (use i = 2 and j = 1)

$J_2 = -\ \{(\partial_1^2 u)(\partial_2 v) - (\partial_2\partial_1^2 u) v\}$ $\qquad$ i > j $\quad$ i = 2, j = 1

$J_1 = -\{ u(\partial_1\partial_2^2 v) - (\partial_1 u)(\partial_2^2 v)\}$ $\qquad$ i > j

$-\mathbf{J}(u,v) = \{(\partial_1^2 u)(\partial_2 v) - (\partial_2\partial_1^2 u) v\}\ \hat{\mathbf{2}} + \{ u(\partial_1\partial_2^2 v) - (\partial_1 u)(\partial_2^2 v)\}\ \hat{\mathbf{1}}$

Well, really it is div J(u,v) that must have the sign change property, not J(u,v). So let's compute that div:

$$\nabla\bullet(-\mathbf{J}(u,v)) = \partial_1\ \{ u(\partial_1\partial_2^2 v) - (\partial_1 u)(\partial_2^2 v)\} + \partial_2\ \{(\partial_1^2 u)(\partial_2 v) - (\partial_2\partial_1^2 u) v\}$$

$$= (\partial_1 u)(\partial_1\partial_2^2 v) + u(\partial_1^2\partial_2^2 v) - (\partial_1^2 u)(\partial_2^2 v) - (\partial_1 u)(\partial_1\partial_2^2 v)$$

$$+ (\partial_2\partial_1^2 u)(\partial_2 v) + (\partial_1^2 u)(\partial_2^2 v) - (\partial_2^2\partial_1^2 u) v - (\partial_2\partial_1^2 u)(\partial_2 v)$$

$$= + u(\partial_1^2\partial_2^2 v) - v(\partial_2^2\partial_1^2 u)$$

and this DOES have the required symmetry.

Plan B for debug (earlier). Suppose we had just $L = \partial_1\partial_2$. This L is formally self adjoint, so I would expect to have J(v,u) = - J(u,v) . Let's compute things from scratch:

$$J_s = -(-1)^{\overset{\text{up}}{ks\ldots+kn}} \Sigma_{m=0}^{ks-1} (-1)^m$$
$$(\partial_s^m\ \underset{\text{down}}{[\partial_{s-1}^{k(s-1)}\ldots\partial_2^{k2}\partial_1^{k1} u]})\ \ (\partial_s^{ks-m-1}\ \underset{\text{up}}{[\partial_{s+1}^{k(s+1)}\ \ldots.\ \partial_n^{kn}(a_{k1k2\ldots kn} v)]})$$

We have $k_1 = 1$ and $k_2 = 1$. So our formula above gives:

$$J_1 = -(-1)^2 \Sigma_{m=0}^{0} (-1)^m$$
$$(\partial_1^m\ [u])\ \ (\partial_1^{-m}[\partial_2(v)]) = -u(\partial_2 v)$$

$$J_2 = -(-1)^1 \Sigma_{m=0}^{1-1} (-1)^m$$
$$(\partial_2^m\ [\partial_1 u])\ \ (\partial_2^{-m}[(v)]) = +(\partial_1 u) v$$

So we seem to get

$$\mathbf{J} = (\partial_1 u) v\ \hat{\mathbf{2}} - u(\partial_2 v)\ \hat{\mathbf{1}}$$

$$\nabla\bullet\mathbf{J} = \partial_1[-u(\partial_2 v)] + \partial_2[(\partial_1 u) v]$$

$= - (\partial_1 u)\ (\partial_2 v) - u(\partial_1\partial_2 v)\ + (\partial_2\partial_1 u)v\ + (\partial_1 u)\ (\partial_2 v)$

$= - u(\partial_1\partial_2 v)\ + v\ (\partial_2\partial_1 u)$

This result DOES have the proper symmetry, so Plan B does not reveal the problem.

****

No need to go below this line till the above problem is resolved. // Problem is resolved, so we can now continue.

Resume after doing symmetry tests, all of which passed. We got to this point in our ∇^4 calculation:

$$- \mathbf{J} = \Sigma_{i=1}^{n}\ \hat{\mathbf{i}}\ \{ u\ (\partial_i^3 v) - (\partial_i u)\ (\partial_i^2 v) + (\partial_i^2 u)\ (\partial_i v) - (\partial_i^3 u)\ v \}$$

$$+ 2\ \Sigma_{i=1}^{n}\ \hat{\mathbf{i}}\ \Sigma_{j=1}^{i-1} [\ (\partial_j^2 u)\ (\partial_i v)\ -\ (\partial_i\partial_j^2 u)\ v\ ]$$

$$+ 2\ \Sigma_{j=1}^{n}\ \hat{\mathbf{j}}\ \Sigma_{i=j+1}^{n} [\ u\ (\partial_j\partial_i^2 v)\ -(\partial_j u)\ (\partial_i^2 v\ )\ ]$$

Swap dummy indices in the last term

$$- \mathbf{J} = \Sigma_{i=1}^{n}\ \hat{\mathbf{i}}\ \{ u\ (\partial_i^3 v) - (\partial_i u)\ (\partial_i^2 v) + (\partial_i^2 u)\ (\partial_i v) - (\partial_i^3 u)\ v \}$$

$$+\ \Sigma_{i=1}^{n}\ \hat{\mathbf{i}}\ \Sigma_{j=1}^{i-1} [2\ (\partial_j^2 u)\ (\partial_i v)\ -\ 2\ (\partial_i\partial_j^2 u)\ v\ ]$$

$$+\ \Sigma_{i=1}^{n}\ \hat{\mathbf{i}}\ \Sigma_{j=i+1}^{n} [2u\ (\partial_i\partial_j^2 v)\ -2\ (\partial_i u)\ (\partial_j^2 v\ )\ ]$$

Now write as a single sum:

I guess we can read off the components of the current as follows:

$$-J_i = \{ u\ (\partial_i^3 v) - (\partial_i u)\ (\partial_i^2 v) + (\partial_i^2 u)\ (\partial_i v) - (\partial_i^3 u)\ v \}$$

$$+ 2\Sigma_{j=1}^{i-1} [(\partial_j^2 u)\ (\partial_i v)\ -\ (\partial_i\partial_j^2 u)\ v\ ]$$

$$+ 2\Sigma_{j=i+1}^{n} [u\ (\partial_i\partial_j^2 v)\ -(\partial_i u)\ (\partial_j^2 v\ )\ ]$$

Now, the answer given on page 40 is supposed to be this:

$$- J_i = - v\ \partial_i\ \nabla^2 u\ + u\ \partial_i\ \nabla^2 v - \nabla^2 v\ \partial_i u\ +\ \nabla^2 u\ \partial_i v$$

Do these two forms agree? First of all, recall that

$$\nabla \bullet (\nabla \times \mathbf{A}) = 0$$

so it is possible that my **J** and Stakgold's **J** could differ by the curl of some vector. What might such a vector look like in this situation?

$$(\nabla \times \mathbf{A})_1 = \quad \varepsilon_{1mn}\partial_m A_n = \partial_2 A_3 - \partial_3 A_2$$

There are not that many bilinear vectors you can construct, one would be this

$$A_i = u\,(\partial_i v)$$

but I would think we would have to have some form of ε_{ijk} kicking around somewhere to make this fact be true. So this is just something to be aware of!

My problem is how to handle the two sums. Here again is our result:

$$-J_i = \{ u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{i-1} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=i+1}^{n} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

Look at the two j sums:

1 2 3 ..... i-1 i i+1 ..... n
<---- first sum ---> <---second sum --->

One idea is to add and subtract equal terms such that we fill out both these sums. For example we might add and subtract this quantity:

$$2\Sigma_{j=i}^{n} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=1}^{i} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

Doing this, we would end up with

$$-J_i = \{ u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{n} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=1}^{n} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

$$- 2\Sigma_{j=i}^{n} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$- 2\Sigma_{j=1}^{i} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

The only benefit of doing this is that we can now simplify the completed sums:

$$-J_i = \{ u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+2 [(\nabla^2 u)(\partial_i v) - (\partial_i \nabla^2 u) v]$$

$$+ 2[u(\partial_i \nabla^2 v) - (\partial_i u)(\nabla^2 v)]$$

$$- 2\Sigma_{j=i}^{n} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$- 2\Sigma_{j=1}^{i} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

so now we have caused some ∇^2 objects to appear. I don't think this helps much, so go back to:

$$-J_i = \{ u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{i-1} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=i+1}^{n} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

Look at the two j sums:

1 2 3 ..... i-1 i i+1 ..... n
<---- first sum ---> <---second sum --->

We can compress things a little as follows:

$$f_i(u,v) \equiv u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v)$$

$$g_{ij}(u,v) \equiv (\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v$$

Then our current is this

$$-J_i = f_i(u,v) - f_i(v,u)$$

$$+ 2\Sigma_{j=1}^{i-1} g_{ij}(u,v)$$

$$- 2\Sigma_{j=i+1}^{n} g_{ij}(v,u)$$

but how does this help me?

Let's try the famous case n = 2 and just write things out. Go back to:

$$-J_i = \{ u(\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{i-1} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=i+1}^{2} [u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

Set i = 2 and get

$$-J_2 = \{ u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v) + (\partial_2^2 u)(\partial_2 v) - (\partial_2^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{1} [(\partial_j^2 u)(\partial_2 v) - (\partial_2 \partial_j^2 u) v]$$

$$-J_2 = u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v) + (\partial_2^2 u)(\partial_2 v) - (\partial_2^3 u) v$$

$$+ 2(\partial_1^2 u)(\partial_2 v) - 2(\partial_2 \partial_1^2 u) v$$

Combine terms where possible

$$-J_2 = u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v)$$

$$+ (2\partial_1^2 u + \partial_2^2 u)(\partial_2 v) - (\partial_2[2(\partial_1^2 u) + \partial_2^2 u]) v$$

$$-J_2 = u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v)$$

$$+ (\partial_1^2 u + \partial_2^2 u)(\partial_2 v) - (\partial_2[(\partial_1^2 u) + \partial_2^2 u]) v$$

$$(\partial_1^2 u)(\partial_2 v) - (\partial_2[(\partial_1^2 u)) v$$

$$-J_2 = u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v)$$

$$+ (\nabla^2 u)(\partial_2 v) - (\partial_2[\nabla^2 u]) v$$

$$(\partial_1^2 u)(\partial_2 v) - (\partial_2 \partial_1^2 u) v$$

$$-J_2 = u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v) + (\partial_1^2 u)(\partial_2 v) - (\partial_2 \partial_1^2 u) v$$

$$+ (\nabla^2 u)(\partial_2 v) - (\partial_2[\nabla^2 u]) v$$

Now let's just confirm or deny whether this agrees with Stak p 40. He gets:

$$-J_2 = -v(\partial_2 \nabla^2 u) + u(\partial_2 \nabla^2 v) - \nabla^2 v (\partial_2 u) + \nabla^2 u (\partial_2 v)$$

In order to have agreement, the following would have to be true:

$$u(\partial_2^3 v) - (\partial_2 u)(\partial_2^2 v) + (\partial_1^2 u)(\partial_2 v) - (\partial_2 \partial_1^2 u) v$$

$= u\,(\partial_2\,\nabla^2 v) - \nabla^2 v\;\;(\partial_2 u)$

Write out the second line in full

$u\,(\partial_2{}^3 v) - (\partial_2\,u)\,(\partial_2{}^2 v) + (\partial_1{}^2\,u)\,(\partial_2 v) \;\; - \;\; (\partial_2\partial_1{}^2\,u)\,v$

$= u\,(\partial_2\,[\partial_1{}^2 + \partial_2{}^2]\,v) - \;([\partial_1{}^2 + \partial_2{}^2]v)\;\;(\partial_2 u)$

Expand, then cancel equal terms

$u\,(\partial_2{}^3 v) - (\partial_2\,u)\,(\partial_2{}^2 v) + (\partial_1{}^2\,u)\,(\partial_2 v) \;\; - \;\; (\partial_2\partial_1{}^2\,u)\,v$

$= u\,(\partial_2\,\partial_1{}^2\,v) + (u\,\partial_2{}^3 v) \;\; - \;\; (\partial_1{}^2\,v)\;\;(\partial_2 u) - \;\; (\partial_2{}^2 v)\,(\partial_2 u)$

$(\partial_1{}^2\,u)\,(\partial_2 v) \;\; - \;\; (\partial_2\partial_1{}^2\,u)\,v$

$= u\,(\partial_2\,\partial_1{}^2\,v) \;\; - \;\; (\partial_1{}^2\,v)\;\;(\partial_2 u)$

Rewrite as

$[\;(\partial_1{}^2\,u)\,(\partial_2 v) + (\partial_1{}^2\,v)\;\;(\partial_2 u) \;\; - \;\; u\,(\partial_1{}^2\,\partial_2\,v) - \;\; (\partial_1{}^2\,\partial_2 u)\;\;v\;] \;\; = 0$

Consider this possibility:

$\partial_1\;\{\;\;(\partial_1\,u)\,(\partial_2 v) + \;\;(\partial_1\,v)\;\;(\partial_2 u) - u\,(\partial_1\,\partial_2\,v) - \;\;(\partial_1\,\partial_2 u)\;\;v\;\;\}$

$= [\;(\partial_1{}^2\,u)\,(\partial_2 v) + (\partial_1{}^2\,v)\;\;(\partial_2 u) \;\; - \;\; u\,(\partial_1{}^2\,\partial_2\,v) - \;\; (\partial_1{}^2\,\partial_2 u)\;\;v]$

$+ \;\;(\partial_1\,u)\,(\partial_1\partial_2 v) + (\partial_1\,v)\;\;(\partial_1\partial_2 u) \;\; - (\partial_1 u)\,(\partial_1\,\partial_2\,v) - \;\;(\partial_1\,\partial_2 u)\,(\partial_1 v)$

$= \;\;[\;(\partial_1{}^2\,u)\,(\partial_2 v) + (\partial_1{}^2\,v)\;\;(\partial_2 u) \;\; - \;\; u\,(\partial_1{}^2\,\partial_2\,v) - \;\; (\partial_1{}^2\,\partial_2 u)\;\;v\;] \;\; + \;\; 0$

So amazingly we can write our required equality in this manner

$-\,\partial_1\;\{\;\;(\partial_1\,u)\,(\partial_2 v) + \;\;(\partial_1\,v)\;\;(\partial_2 u) - u\,(\partial_1\,\partial_2\,v) - \;\;(\partial_1\,\partial_2 u)\;\;v\;\;\} \;\; = 0$

I know that if I make the vector $\mathbf{A} = (\partial_2 f, -\partial_1 f)$, then $\nabla\bullet\mathbf{A} = 0$ in 2D. So maybe we can think of the item in {...} above as the object "f". Then it is *possible* that my and Stak's currents differ by a vector which has zero divergence in 2D. To confirm this, I need to now compute J_1 ! But let's not bother with that!

Plan C for doing ∇^4 . Let's try to "be more symmetric". Go back to this point:

$$J_i = - \{(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v\} \qquad i > j$$

$$J_j = - \{ u (\partial_j \partial_i^2 v) - (\partial_j u)(\partial_i^2 v)\} \qquad i > j$$

$$J_i = - \{u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)\} \qquad i < j$$

$$J_j = - \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u)(v) \} \qquad i < j$$

$$J_i = - \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \} \qquad i = j$$

Then write L in this more symmetric way:

$$L = \Sigma_{i=1}^{n} \partial_i^4 + \Sigma_{i>j} \partial_i^2 \partial_j^2 + \Sigma_{j>i} \partial_i^2 \partial_j^2$$

This is just showing a series of terms that makes up L. For each term we have a current. So the total current is then this:

$$-\mathbf{J} = \Sigma_{i=1}^{n} \hat{\mathbf{i}} \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ \Sigma_{i>j} [\hat{\mathbf{i}} \{(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v\} + \hat{\mathbf{j}} \{ u (\partial_j \partial_i^2 v) - (\partial_j u)(\partial_i^2 v)\}]$$

$$+ \Sigma_{j>i} [\hat{\mathbf{i}} \{u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)\} + \hat{\mathbf{j}} \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u)(v) \}]$$

This is my "more symmetric form", but I don't think it has bought me anything. I suppose I could now do swaps in both second terms to get this form:

$$-\mathbf{J} = \Sigma_{i=1}^{n} \hat{\mathbf{i}} \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ \Sigma_{i>j} \hat{\mathbf{i}} \{(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v\} + \Sigma_{i>j} \hat{\mathbf{j}} \{ u (\partial_j \partial_i^2 v) - (\partial_j u)(\partial_i^2 v)\}$$

$$+ \Sigma_{j>i} \hat{\mathbf{i}} \{u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)\} + \Sigma_{j>i} \hat{\mathbf{j}} \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u)(v) \}$$

At this point, swap indices on both last lines on the right,

$$-\mathbf{J} = \Sigma_{i=1}^{n} \hat{\mathbf{i}} \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ \Sigma_{i>j} \hat{\mathbf{i}} \{(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v\} + \Sigma_{i<j} \hat{\mathbf{i}} \{ u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)\}$$

$$+ \Sigma_{j>i} \hat{\mathbf{i}} \{u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)\} + \Sigma_{i>j} \hat{\mathbf{i}} \{ (\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u)(v) \}$$

But now sums are pairwise the same, so get

$$-\mathbf{J} = \Sigma_{i=1}^{n} \hat{\mathbf{i}} \{ u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v) + (\partial_i^2 u) (\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2 \Sigma_{i>j} \hat{\mathbf{i}} \{(\partial_j^2 u) (\partial_i v) - (\partial_i \partial_j^2 u) v\}$$

$$+ 2 \Sigma_{j>i} \hat{\mathbf{i}} \{u (\partial_i \partial_j^2 v) - (\partial_i u) (\partial_j^2 v) \}$$

I don't think I have anything new here. As before, rewrite as

$$-\mathbf{J} = \Sigma_{i=1}^{n} \hat{\mathbf{i}} \{ u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v) + (\partial_i^2 u) (\partial_i v) - (\partial_i^3 u) v \}$$

$$+ \Sigma_{i=1}^{n} \hat{\mathbf{i}} \; 2 \Sigma_{j<i} \{(\partial_j^2 u) (\partial_i v) - (\partial_i \partial_j^2 u) v\}$$

$$+ \Sigma_{i=1}^{n} \hat{\mathbf{i}} \; 2 \Sigma_{j>i} \{u (\partial_i \partial_j^2 v) - (\partial_i u) (\partial_j^2 v) \}$$

I would then pick off the $-J_i$ component as follows:

$$-J_i = \{ u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v) + (\partial_i^2 u) (\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2 \Sigma_{j<i} \{(\partial_j^2 u) (\partial_i v) - (\partial_i \partial_j^2 u) v\}$$

$$+ 2 \Sigma_{j>i} \{u (\partial_i \partial_j^2 v) - (\partial_i u) (\partial_j^2 v) \}$$

Yes, I have still maintained some symmetry, but this is the same result I got earlier. Let's again compactify exactly as before:

$$f_i(u,v) \equiv u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v)$$

$$g_{ij}(u,v) \equiv (\partial_j^2 u) (\partial_i v) - (\partial_i \partial_j^2 u) v$$

$$-J_i(u,v) = f_i(u,v) - f_i(v,u) + 2 \Sigma_{j<i} \; g_{ij}(u,v) - 2 \Sigma_{j>i} g_{ij}(v,u)$$

Another thing I could do is move in the sums from the previous set

$$-J_i = \{ u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v) + (\partial_i^2 u) (\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2 \Sigma_{j<i} (\partial_j^2 u) (\partial_i v) - 2 \Sigma_{j<i} (\partial_i \partial_j^2 u) v$$

$$+ 2 \Sigma_{j>i} u (\partial_i \partial_j^2 v) - 2 \Sigma_{j>i} (\partial_i u) (\partial_j^2 v)$$

$$-J_i = \{ u (\partial_i^3 v) - (\partial_i u) (\partial_i^2 v) + (\partial_i^2 u) (\partial_i v) - (\partial_i^3 u) v \}$$

$$+\ 2\,(\partial_i v)\ \Sigma_{j<i}\,(\partial_j^2 u)\ \ -\ 2\,v\ \partial_i\,(\Sigma_{j<i}\,(\partial_j^2 u))$$

$$+\ 2\,u\,\partial_i\,(\Sigma_{j>i}\,(\partial_j^2 v))\ \ -\ 2\,(\partial_i u)\,\Sigma_{j>i}\,(\partial_j^2 v)$$

$$-J_i\ = \{\ u\,(\partial_i^3 v) - (\partial_i u)\,(\partial_i^2 v) + (\partial_i^2 u)\,(\partial_i v) - (\partial_i^3 u)\,v\ \}$$

$$+\ 2\,[\ \{(\partial_i v) - v\,\partial_i\ \}\{\ \Sigma_{j<i}\,(\partial_j^2 u)\}\ \ -\ 2\,[\ \{(\partial_i u) - u\,\partial_i\ \}\{\ \Sigma_{j>i}\,(\partial_j^2 v)\}$$

Now we have two "partial Laplacians" sitting in this thing. I am always tempted to "fill them out" by adding and subtracting equal terms, but I know that leads me nowhere.

Now do it the simple Stakgold way:

Comments: My form for the current has these ugly partial sums, whereas his form for the current has only "full sums". Suppose we just start over on this specific problem manually as follows:

$$\int dV\ (vLu - uL^*v)\ = \int dV\ \{\ v\,\nabla^2\nabla^2 u\ - u\,\nabla^2\nabla^2 v\ \}$$

$$= \Sigma_{ij} \int dV\ \{\ v\,(\partial_i^2\partial_j^2 u)\ - u\,(\partial_i^2\partial_j^2 v)\ \}\ =\ \Sigma_{ij}\,[\ T1 - T2\ ]$$

Here is my little theorem for shifting things over specialized to k = 2

$$\int_V dV\ \psi(\partial_i^2\varphi)\ =\ \int_V dV\ (\partial_i^2\psi)\ \varphi\ \ +\ \int_S d\mathbf{S}\bullet\hat{\mathbf{i}}\ \ \{\ \Sigma_{m=0}^{1}\,(-1)^m\,(\partial_i^m\psi)\,(\partial_i^{1-m}\varphi)\ \ \}$$

So set $\psi = u$ and $\varphi = (\partial_j^2 v)$ and write

$$T2 = \int dV\ u\,(\partial_i^2\partial_j^2 v)\ =\ \ \int_V dV\ (\partial_i^2 u)\,(\partial_j^2 v)$$

$$+\ \int_S d\mathbf{S}\bullet\hat{\mathbf{i}}\ \ \{\ \Sigma_{m=0}^{1}\,(-1)^m\,(\partial_i^m u)\,(\partial_i^{1-m}(\partial_j^2 v))\ \ \}\ \ =\ T21 + T22$$

Aside: the current so far is $\mathbf{J} = -\,\hat{\mathbf{i}}\ \{\ u\,(\partial_i\,\partial_j^2 v)\ - (\partial_i u)\,(\partial_j^2 v)\ \}$ ***

Now process T21 further:

$$T21 = \int_V dV\ (\partial_i^2 u)\,(\partial_j^2 v)$$

Our theorem this time will be

$$\int_V dV\, \psi(\partial_j^2\varphi) = \int_V dV\, (\partial_j^2\psi)\, \varphi + \int_S d\mathbf{S}\bullet\hat{\mathbf{j}}\ \{ \Sigma_{m=0}^{1} (-1)^m (\partial_j^m\psi)(\partial_j^{1-m}\varphi) \}$$

and this time set $\psi = (\partial_i^2 u)$ and $\varphi = v$. Then we get

$$T21 = \int_V dV\, (\partial_i^2 u)(\partial_j^2 v) = \int_V dV\, (\partial_j^2(\partial_i^2 u))\, v$$

$$+ \int_S d\mathbf{S}\bullet\hat{\mathbf{j}}\ \{ \Sigma_{m=0}^{1} (-1)^m (\partial_j^m(\partial_i^2 u))(\partial_j^{1-m} v) \}$$

Aside: the current from this phase is $\mathbf{J} = -\hat{\mathbf{j}}\ \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u) \}$ ****

We now have achieved cancellation of the main terms, and we have these two current contributions (where the minus sign on the left comes from the fact it was a - T2 to start with)

$$-\mathbf{J} = \hat{\mathbf{i}}\ \{ \Sigma_{m=0}^{1} (-1)^m (\partial_i^m u)(\partial_i^{1-m}(\partial_j^2 v)) \} + \hat{\mathbf{j}}\ \{ \Sigma_{m=0}^{1} (-1)^m (\partial_j^m(\partial_i^2 u))(\partial_j^{1-m} v) \}$$

$$= \hat{\mathbf{i}}\ \{ u(\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v) \} + \hat{\mathbf{j}}\ \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u) \}$$

where we have not yet shown the Σ_{ij} sum . Notice that this is the "total current" due to both flip overs, and the result is independent of whether i > j or i < j. The result even true for i = j, you just treat the two flips separately.

Simplify these terms first

$$-\mathbf{J} = \hat{\mathbf{i}}\ \{ u\, \partial_i \partial_j^2 v - (\partial_i u)(\partial_j^2 v) \} + \hat{\mathbf{j}}\ \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u) v \}$$

Now I think the final answer for the current should be this:

$$-\mathbf{J} = \Sigma_{ij}\ [\ \hat{\mathbf{i}}\ \{ u\, \partial_i \partial_j^2 v - (\partial_i u)(\partial_j^2 v) \} + \hat{\mathbf{j}}\ \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u) v \}\ ]$$

$$-\mathbf{J} = \Sigma_{ij}\ [\ \hat{\mathbf{i}}\ \{ u\, \partial_i \partial_j^2 v - (\partial_i u)(\partial_j^2 v) \}\ ] + \Sigma_{ij}\ [\ \hat{\mathbf{j}}\ \{ (\partial_i^2 u)(\partial_j v) - (\partial_j \partial_i^2 u) v \}\ ]$$

In the first term, move the Σ_j to the right to create ∇^2 objects. In the second term move the Σ_i ahead to get the same thing. Then we have this:

$$-\mathbf{J} = \Sigma_i\ [\ \hat{\mathbf{i}}\ \{ u(\partial_i \nabla^2 v) - (\partial_i u)(\nabla^2 v) \}\ ] + \Sigma_j\ [\ \hat{\mathbf{j}}\ \{ (\nabla^2 u)(\partial_j v) - (\partial_j \nabla^2 u) v \}\ ]$$

Now in the second term, replace the dummy change the dummy index from j to i

$$-\mathbf{J} = \Sigma_i\ [\ \hat{\mathbf{i}}\ \{ u(\partial_i \nabla^2 v) - (\partial_i u)(\nabla^2 v) \}\ ] + \Sigma_i\ [\ \hat{\mathbf{i}}\ \{ (\nabla^2 u)(\partial_i v) - (\partial_i \nabla^2 u) v \}\ ]$$

Now combine the two sums

$$-\mathbf{J} = \Sigma_i \, \hat{\mathbf{i}} \, \{ u \, (\partial_i \nabla^2 v) \; - (\partial_i u) \, (\nabla^2 v) \; + (\nabla^2 u)(\partial_i v) - (\partial_i \nabla^2 u) v \; \}$$

Then conclude that

$$-J_i = u \, (\partial_i \nabla^2 v) \; - (\partial_i u) \, (\nabla^2 v) \; + (\nabla^2 u)(\partial_i v) - (\partial_i \nabla^2 u) v$$

which means

$$-J = u \, (\text{grad} \, \nabla^2 v) \; - (\text{grad} \, u) \, (\nabla^2 v) \; + (\nabla^2 u)(\text{grad} \, v) - (\text{grad} \, \nabla^2 u) v$$

which agrees exactly with Stak page 40. So finally I have at least obtained his result.