Various Meanings of "Green's Theorem" PhL 6.30.12

The phrase Green's Theorem seems to mean different things to different people. I will try here to make a short list.

1. First, we at least state what are usually called the two **Green's Identities**:

$$\int_V dV\, \psi\, \nabla^2\varphi = \int_S dS\, \psi(\partial\varphi/\partial n) - \int_V dV\, \{\nabla\psi \bullet \nabla\varphi\} \qquad (5a) \quad \text{Green \#1}$$

$$\int_V dV\, [\, \psi\, \nabla^2\varphi - \varphi\, \nabla^2\psi\,] = \int_S dS\, [\, \psi(\partial\varphi/\partial n) - \varphi(\partial\psi/\partial n)\,] \qquad (6) \quad \text{Green \#2}$$

$$= \int_S d\mathbf{S} \bullet [\, \psi\nabla\varphi - \varphi\nabla\psi\,]$$

Do people refer to either of these as "Green's Theorem". Yes!

On page 14 green **Jackson** refers to the 2nd identity both by that name, *and* by the term Green's Theorem.

And on p 161 **M&M** also refer to *both* of the above as "Green's Theorems".

And Chap 7 p 806 **M&F** also refer to Green #2 as Green's Theorem.

2. **Stakgold** seems to have a special definition of "Green's Theorem" in his 5.72 and 5.73 p 40

$$vLu - uL^*v = \text{div}\, \mathbf{J} \qquad (5.72)$$

$$\int_V dV\, (vLu - uL^*v) = \int_S dS\ \mathbf{n} \bullet \mathbf{J} \qquad (5.73)$$

So in Stak's version, L can be any linear differential operator! And of course the space holding dV can have any number of dimensions N. So his I think is the most general. If you set $L = \nabla^2$, then we know from Stak page 40 that $\mathbf{J} = u\, \nabla\, v - v\, \nabla\, u$ and in that case Stak's Green's Theorem becomes the second identity. So really Stak's thing is a generalization of the second identity.

3. **Lai** on page 411 uses the term "Green's Theorem" to describe the "integral of a gradient theorem" in 2D space. That is to say, from my sheet [I think this requires that C be "convex"]

$$\int_V dV\, \nabla\varphi = \int_S d\mathbf{S}\, \varphi = \int_S dS\ \mathbf{n}\, \varphi \qquad \text{// "integral of a gradient theorem"}$$

and in 2D this becomes

$$\int_A dA\partial_x\varphi = \int_C ds\, n_x\, \varphi = \int_C dy\, \varphi$$

$$\int_A dA\partial_y\varphi = \int_C ds\, n_y\, \varphi = -\int_C dx\, \varphi$$

4. **Wiki**: Now suppose we consider the 1st line above applied to $\varphi = M$ and the 2nd line applied to $\varphi = L$,

$$\int_A dA\partial_x M = \int_C ds\, n_x M = \int_C dy\, M$$

$$\int_A dA\partial_y L = \int_C ds\, n_y L = -\int_C dx\, L$$

Now subtract the second line from the first to get

$$\int_A dA\,(\partial_x M - \partial_y L) = \int_C dy\, M + \int_C dx\, L = \int_C (dy\, M + dx\, L)$$

According to wiki, this is **yet another "Green's Theorem"** that differs from all of the above!!

Theorem [edit]

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane $\mathbb{R}^2$, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then[1][2]

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy.$$

For positive orientation, an arrow pointing in the counterclockwise direction may be drawn in the small circle in the integral symbol.

In physics, Green's theorem is mostly used to solve two-dimensional flow integrals, stating that the sum of fluid outflows at any point inside a volume is equal to the total outflow summed about an enclosing area. In plane geometry, and in particular, area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.

Divergence theorem

Multivariable calculus [show]

V• T• E

Suppose the two functions form a vector of functions

$$M = f_y \qquad L = f_x$$

then the above theorem becomes

$$\int_A dA\,(\partial_x f_y - \partial_y f_x) = \int_C dy\, f_y + \int_C dx\, f_x = \int_C (dy\, f_y + dx\, f_x) = \int_C d\mathbf{x} \bullet f$$

My PDF called "chap3 integral theorems" refers to this as "**Green's Theorem in a Plane**" and also "**Stokes Theorem in a Plane**".