Parts Integration and the Green's Theorem Current PhL 7.8.09

Some items claimed here are proven in Stak Chap 5 raw notes "attempts to prove the full p 40 Greens..." . Those raw notes are a big mess, but parts are correct (those parts referred to in this doc).

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1. Single parts integration of order 1 in R_n

On my Integral Theorems page, I have this confirmed vector theorem [it is easily derived as a simple corollary to the divergence theorem, see integral theorems.doc]

$$\int dV\ \psi\ (\nabla\varphi) = -\int dV\ (\nabla\psi)\ \varphi\ + \int d\mathbf{S}\ (\psi\varphi) \qquad \text{// 2a, "parts integration theorem"}$$

It is perhaps simpler to think of a particular component of this theorem,

$$\int dV\ \psi\ (\partial_i\varphi) = -\int dV\ (\partial_i\psi)\ \varphi\ + \int d\mathbf{S} \bullet \hat{\mathbf{i}}\ (\psi\varphi)$$

We could have just put dS_i on the right instead of $d\mathbf{S} \bullet \hat{\mathbf{i}}$, but this latter form lets us state the result in terms of what I like to call a "current" **J**. We restate the above then as:

$$\int dV\ \psi\ (\partial_i\varphi) = -\int dV\ (\partial_i\psi)\ \varphi\ + \int d\mathbf{S} \bullet \mathbf{J} \qquad \text{where} \qquad \mathbf{J} = \hat{\mathbf{i}}\ (\psi\varphi) \qquad J_i = \psi\varphi$$

We are in R_n space here, and the above is the generalization of the familiar "parts integration" in 1 dimension. Here, "the parts" is the surface integral of the current shown. If we had the differential operator $L = \Sigma_i\partial_i = d/dx + d/dy + ...$, then we would add up the currents to get $\mathbf{J} = \psi\varphi(1,1,1.....)$, so in this simple case, all the current components would be the same.

2. Single parts integration of order k in R_n

It is not hard to generalize the above result for the case $\partial_i \rightarrow \partial_i{}^k$. I have derived this elsewhere, using an induction proof, and here is the result: (see "attempts to prove the full p 40....doc" part C, this needs to be cleaned up!!!!)

Theorem:

$$\int dV\ \psi(\partial_i{}^k\varphi) = (-1)^k \int dV\ (\partial_i{}^k\psi)\ \varphi + \int d\mathbf{S}\bullet\ \mathbf{J} \text{ where } \mathbf{J} = \hat{\mathbf{i}}\ \{\ \Sigma_{m=0}{}^{k-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k-m-1}\varphi)\ \}$$

As an example, we can set k = 2 to get $\mathbf{J} = \hat{\mathbf{i}}\ \{\ \Sigma_{m=0}{}^{1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{1-m}\varphi)\ \}$ so that

$$\int dV\ \psi(\partial_i{}^2\varphi) = \int dV\ (\partial_i{}^2\psi)\ \varphi + \int d\mathbf{S}\bullet\ \mathbf{J} \quad \text{where } \mathbf{J} = \hat{\mathbf{i}}\ \{\psi\ (\partial_i\varphi) - (\partial_i\psi)\ \varphi\ \}$$

If we had $L = \Sigma_i\ \partial_i{}^2$ ($= \nabla^2$), then for each term we get a current J_i and we add them up to get the total current. In this case, we can write $J_i = \{\psi\ (\partial_i\varphi) - (\partial_i\psi)\ \varphi\ \}$ and $\mathbf{J} = \psi(\nabla\varphi) - (\nabla\psi)\varphi$. Notice that, unlike our previous example with k = 1, the components of the current are now in general *different* since ψ and φ are arbitrary functions over R_n.

3. Two parts integrations $\partial_i{}^{k1}\partial_j{}^{k2}$: non-uniqueness of the surface current J

Suppose we have two partials instead of 1, each with a power. We might for example be interested in:

$$\int dV\ \psi(\partial_i{}^{k1}\partial_j{}^{k2}\varphi)$$

where I used k_1 and k_2 (written out flat) as the order of each derivative, and i and j refer to two components in R_n space. In R_3 we might have i = 1 and j = 2. To handle this "double parts" operation, we will use our theorem of Part 2 twice. The first time our theorem states this, where we replace k with k1 and φ with θ

$$\int dV\ \psi(\partial_i{}^{k1}\theta) = (-1)^{k1} \int dV\ (\partial_i{}^{k1}\psi)\ \theta + \int d\mathbf{S}\bullet\ \hat{\mathbf{i}}\ \{\ \Sigma_{m=0}{}^{k1-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k1-m-1}\theta)\ \}$$

but we now of course want to set $\theta = (\partial_j{}^{k2}\varphi)$. We then obtain this intermediate result:

$$\int dV\ \psi(\partial_i{}^{k1}\partial_j{}^{k2}\varphi) = (-1)^{k1} \int dV\ (\partial_i{}^{k1}\psi)\ (\partial_j{}^{k2}\varphi)$$

$$+ \int d\mathbf{S}\bullet\ \hat{\mathbf{i}}\ \{\ \Sigma_{m=0}{}^{k1-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ \}$$

Next, we shall process the first integral on the right with another version of our theorem:

$$\int dV\ \alpha\ (\partial_j{}^{k2}\beta) = (-1)^{k2} \int dV\ (\partial_j{}^{k2}\alpha\ )\ \beta + \int d\mathbf{S}\bullet\ \hat{\mathbf{j}}\ \{\ \Sigma_{m=0}{}^{k2-1}\ (-1)^m\ (\partial_j{}^m\ \alpha)\ (\partial_j{}^{k2-m-1}\beta)\ \}$$

where we now set $k = k_2$ and i = j. Obviously we now want to set $\alpha = (\partial_i{}^{k1}\psi)$ and $\beta = \varphi$ so we have

$$\int dV\ (\partial_i{}^{k1}\psi)\ (\partial_j{}^{k2}\varphi)$$

$$= (-1)^{k2} \int dV\ (\partial_j{}^{k2}\partial_i{}^{k1}\psi)\ \varphi \ + \int d\mathbf{S}\bullet \hat{\mathbf{j}}\ \{ \Sigma_{m=0}{}^{k2-1}\ (-1)^m\ (\partial_j{}^m\ \ (\partial_i{}^{k1}\psi))\ (\partial_j{}^{k2-m-1}\ \varphi)\ \}$$

Putting the pieces together, we have now shown that

$$\int dV\ \psi(\partial_i{}^{k1}\partial_j{}^{k2}\varphi) = (-1)^{k1+k2} \int dV\ (\partial_j{}^{k2}\partial_i{}^{k1}\psi)\ \varphi$$

$$+ \int d\mathbf{S}\bullet \hat{\mathbf{i}}\ \{ \Sigma_{m=0}{}^{k1-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ \}$$

$$+ (-1)^{k1}\ \int d\mathbf{S}\bullet \hat{\mathbf{j}}\ \{ \Sigma_{m=0}{}^{k2-1}\ (-1)^m\ (\partial_j{}^m\partial_i{}^{k1}\psi)\ (\partial_j{}^{k2-m-1}\ \varphi)\ \}$$

We can combine the two surface integral integrands into the following single current

$$\mathbf{J} = \hat{\mathbf{i}}\ \ \Sigma_{m=0}{}^{k1-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)$$

$$+ \hat{\mathbf{j}}\ \ (-1)^{k1}\ \Sigma_{m=0}{}^{k2-1}\ (-1)^m\ (\partial_j{}^m\partial_i{}^{k1}\psi)\ (\partial_j{}^{k2-m-1}\ \varphi)$$

and in terms of this current, we state our double parts theorem:

$$\int dV\ \psi(\partial_i{}^{k1}\partial_j{}^{k2}\varphi) = (-1)^{k1+k2} \int dV\ (\partial_j{}^{k2}\partial_i{}^{k1}\psi)\ \varphi \ + \int d\mathbf{S} \bullet \mathbf{J}$$

We see that, for a particular i and j, our current J has the form $\mathbf{J} = J_i\hat{\mathbf{i}} + J_j\hat{\mathbf{j}}$.

Now an important point: we *could* have done this "double parts" in the reverse order, moving $\partial_j{}^{k2}$ first, and then moving $\partial_i{}^{k1}$. Had we done this, we would obtain the same main integral, but the current in the surface integral would be **different**! To get this new current, we just make the simultaneous swaps $i \leftrightarrow j$ and $k1 \leftrightarrow k2$,

$$\mathbf{J'} = \hat{\mathbf{j}}\ \ \Sigma_{m=0}{}^{k2-1}\ (-1)^m\ (\partial_j{}^m\psi)\ (\partial_j{}^{k2-m-1}\partial_i{}^{k1}\varphi)$$

$$+ \hat{\mathbf{i}}\ \ (-1)^{k2}\ \Sigma_{m=0}{}^{k1-1}\ (-1)^m\ (\partial_i{}^m\partial_j{}^{k2}\psi)\ (\partial_i{}^{k1-m-1}\ \varphi)$$

The current **J** and **J'** really are different. Let's look first at a very simple case: $k1 = k2 = 1$. We find that

$$\mathbf{J} = \hat{\mathbf{i}}\ (\psi)\ (\partial_j\varphi)\ -\ \hat{\mathbf{j}}\ (\partial_i\psi)\ (\varphi)$$

$$\mathbf{J'} = \hat{\mathbf{j}}\ (\psi)\ (\partial_i\varphi)\ \ -\ \hat{\mathbf{i}}\ (\partial_j\psi)\ (\varphi) \qquad \neq \mathbf{J}$$

But the following parts integration forms are both "true"

$$\int dV\ \psi(\partial_i\partial_j\varphi) = \int dV\ (\partial_j\partial_i\psi)\ \varphi + \int d\mathbf{S}\bullet\ \mathbf{J}$$

$$\int dV\ \psi(\partial_i\partial_j\varphi) = \int dV\ (\partial_j\partial_i\psi)\ \varphi + \int d\mathbf{S}\bullet\ \mathbf{J'}$$

The two surface integrals *must be the same*. This sameness converted by the divergence theorem says that

$$\int dV\ \ \nabla\bullet\mathbf{J} = \int dV\ \ \nabla\bullet\mathbf{J'}$$

Since this must be true for any volume, it must be true that $\nabla\bullet\mathbf{J} = \nabla\bullet\mathbf{J'}$. In other words, <u>the currents **J** and **J'** must differ by a vector which has zero divergence!</u> In our example above we have

$$\Delta\mathbf{J} = \mathbf{J} - \mathbf{J'} = \hat{\mathbf{i}}\ \psi(\partial_j\varphi) - \hat{\mathbf{j}}\ (\partial_i\psi)\ \varphi - \hat{\mathbf{j}}\ \psi(\partial_i\varphi) + \hat{\mathbf{i}}\ (\partial_j\psi)\ \varphi$$

$$= \hat{\mathbf{i}}\ [\psi(\partial_j\varphi) + (\partial_j\psi)\ \varphi] - \hat{\mathbf{j}}\ [(\partial_i\psi)\ \varphi + \psi(\partial_i\varphi)]$$

If we take the divergence of $\Delta\mathbf{J}$, we get

$$\nabla\bullet\ \Delta\mathbf{J} = \partial_i\ [\psi(\partial_j\varphi) + (\partial_j\psi)\ \varphi] - \partial_j[(\partial_i\psi)\ \varphi + \psi(\partial_i\varphi)]$$

$$= \partial_i[\psi(\partial_j\varphi) + (\partial_j\psi)\ \varphi] - \partial_j[\psi(\partial_i\varphi) + (\partial_i\psi)\ \varphi]$$

$$= \partial_i[\psi(\partial_j\varphi) + (\partial_j\psi)\ \varphi] - (i\leftrightarrow j)$$

$$= (\partial_i\psi)\ (\partial_j\varphi) + \psi(\partial_i\partial_j\varphi) + (\partial_i\partial_j\psi)\ \varphi + (\partial_j\psi)\ (\partial_i\varphi)$$
$$- (\partial_j\psi)\ (\partial_i\varphi) - \psi(\partial_j\partial_i\varphi) - (\partial_j\partial_i\psi)\ \varphi - (\partial_i\psi)\ (\partial_j\varphi)$$

$$= 0 \qquad \text{due to four pairwise cancellations}$$

We may conclude that our two currents differ by a vector which has zero divergence. That vector is this:

$$\Delta\mathbf{J} = \hat{\mathbf{i}}\ [\psi(\partial_j\varphi) + (\partial_j\psi)\ \varphi] - \hat{\mathbf{j}}\ [(\partial_i\psi)\ \varphi + \psi(\partial_i\varphi)]$$

$$\nabla\bullet\ \Delta\mathbf{J} = 0$$

<u>Aside</u>: It is true that the divergence of the curl of a vector is 0, but there are lots of other vectors one can construct that have zero divergence. Although the above $\Delta\mathbf{J}$ has some curl-like appearance, I think it is a simple example of a divergence-free vector that is not a curl of some other vector. I could be wrong. A constant vector is another example of a divergence free vector (perhaps can write as curl of something).

We can now show that $\nabla\bullet\,\Delta\mathbf{J} = 0$ for general case k1 and k2, but it will require some work. We want to show that these two currents have the same divergence:

$$\mathbf{J} = \hat{\mathbf{i}} \sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^m\psi)(\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)$$

$$+ \hat{\mathbf{j}}\ (-1)^{k1} \sum_{m=0}^{k2-1} (-1)^m (\partial_j{}^m\partial_i{}^{k1}\psi)(\partial_j{}^{k2-m-1}\varphi)$$

$$\mathbf{J'} = \hat{\mathbf{j}} \sum_{m=0}^{k2-1} (-1)^m (\partial_j{}^m\psi)(\partial_j{}^{k2-m-1}\partial_i{}^{k1}\varphi)$$

$$+ \hat{\mathbf{i}}\ (-1)^{k2} \sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^m\partial_j{}^{k2}\psi)(\partial_i{}^{k1-m-1}\varphi)$$

Reminder: the current **J'** is obtained from the current **J** by swapping k1↔k2 and i↔j.

Let's take the divergence of each J and show we get the same thing, at which point we say QED:

$$\nabla\bullet\mathbf{J} = \partial_i[\ \sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^m\psi)(\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ ]$$

$$+\ \partial_j\ [(-1)^{k1} \sum_{m=0}^{k2-1} (-1)^m (\partial_j{}^m\partial_i{}^{k1}\psi)(\partial_j{}^{k2-m-1}\varphi)\ ]$$

$$= \sum_{m=0}^{k1-1} (-1)^m [\ (\partial_i{}^{m+1}\psi)(\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ + (\partial_i{}^m\psi)(\partial_i{}^{k1-m}\partial_j{}^{k2}\varphi)\ ]$$

$$+ (-1)^{k1} \sum_{m=0}^{k2-1} (-1)^m [(\partial_j{}^{m+1}\partial_i{}^{k1}\psi)(\partial_j{}^{k2-m-1}\varphi) + (\partial_j{}^m\partial_i{}^{k1}\psi)(\partial_j{}^{k2-m}\varphi)\ ]$$

$$\nabla\bullet\mathbf{J'} = \partial_j\ [\sum_{m=0}^{k2-1} (-1)^m (\partial_j{}^m\psi)(\partial_j{}^{k2-m-1}\partial_i{}^{k1}\varphi)\ ]$$

$$+ \partial_i\ [(-1)^{k2} \sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^m\partial_j{}^{k2}\psi)(\partial_i{}^{k1-m-1}\varphi)]$$

$$= \sum_{m=0}^{k2-1} (-1)^m [(\partial_j{}^{m+1}\psi)(\partial_j{}^{k2-m-1}\partial_i{}^{k1}\varphi) + (\partial_j{}^m\psi)(\partial_j{}^{k2-m}\partial_i{}^{k1}\varphi)\ ]$$

$$+ (-1)^{k2} \sum_{m=0}^{k1-1} (-1)^m [(\partial_i{}^{m+1}\partial_j{}^{k2}\psi)(\partial_i{}^{k1-m-1}\varphi)\ + (\partial_i{}^m\partial_j{}^{k2}\psi)(\partial_i{}^{k1-m}\varphi)\ ]$$

Observation: $\nabla\bullet\mathbf{J'}$ is obtained from $\nabla\bullet$J by swapping k1↔k2 and i↔j.

There is a **big trick** to doing this. Consider the first line of the $\nabla\bullet\mathbf{J}$ result above which reads:

$$\sum_{m=0}^{k1-1} (-1)^m [\ (\partial_i{}^{m+1}\psi)(\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ + (\partial_i{}^m\psi)(\partial_i{}^{k1-m}\partial_j{}^{k2}\varphi)\ ]$$

First, write this as two separate sums:

$$\sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^{m+1}\psi)(\partial_i{}^{k1-m-1}\partial_j{}^{k2}\varphi)\ +\ \sum_{m=0}^{k1-1} (-1)^m (\partial_i{}^m\psi)(\partial_i{}^{k1-m}\partial_j{}^{k2}\varphi)$$

Expose the m=0 term of the second sum:

$$\Sigma_{m=0}^{k1-1} (-1)^m (\partial_i^{m+1}\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi) \; + \; \Sigma_{m=1}^{k1-1} (-1)^m (\partial_i^{m}\psi) (\partial_i^{k1-m} \partial_j^{k2}\varphi) \; + \psi(\partial_i^{k1} \partial_j^{k2}\varphi)$$

Now introduce m = m'+1 in the second sum, m' = m-1, putting minus sign between terms,

$$\Sigma_{m=0}^{k1-1} (-1)^m (\partial_i^{m+1}\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi) - \Sigma_{m'=0}^{k1-2} (-1)^{m'} (\partial_i^{m'+1}\psi) (\partial_i^{k1-m'-1} \partial_j^{k2}\varphi) \; + \psi(\partial_i^{k1} \partial_j^{k2}\varphi)$$

Now relabel m' to be m:

$$\Sigma_{m=0}^{k1-1} (-1)^m (\partial_i^{m+1}\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi) - \Sigma'_{=0}^{k1-2} (-1)^m (\partial_i^{m+1}\psi) (\partial_i^{k1-m-1} \partial_j^{k2}\varphi) \; + \psi(\partial_i^{k1} \partial_j^{k2}\varphi)$$

Now expose the upper term of the first sum which has m = k1-1

$$(-1)^{k1-1} (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi)$$

The residual sums exactly cancel, and we are left with this result for the first line of $\nabla\bullet\mathbf{J}$

$$\psi(\partial_i^{k1} \partial_j^{k2}\varphi) - (-1)^{k1} (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi)$$

If we carry out this exact same process in the second line of $\nabla\bullet\mathbf{J}$, we are left again with the first term of the second sum, and with the last term of the first sum. Here is that second line

$$+ (-1)^{k1} \Sigma_{m=0}^{k2-1} (-1)^m [(\partial_j^{m+1}\partial_i^{k1}\psi) (\partial_j^{k2-m-1} \varphi) + (\partial_j^{m}\partial_i^{k1}\psi) (\partial_j^{k2-m} \varphi)]$$

The first term of the second sum is the one with m = 0. That term is:

$$(-1)^{k1} \; (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi)$$

The last term of the first sum is the one with m = k2-1. That term is:

$$(-1)^{k1} \; (-1)^{k2-1}(\partial_j^{k2}\partial_i^{k1}\psi) (\varphi)$$

Thus, our second line gives

$$(-1)^{k1} \; (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi) \; + (-1)^{k1} \; (-1)^{k2-1}(\partial_j^{k2}\partial_i^{k1}\psi) (\varphi)$$

We can now combine the first and second lines of $\nabla\bullet\mathbf{J}$ with this result

$$\psi(\partial_i^{k1} \partial_j^{k2}\varphi) - (-1)^{k1} (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi)$$

$$+ (-1)^{k1} \; (\partial_i^{k1}\psi) (\partial_j^{k2}\varphi) \; + (-1)^{k1} \; (-1)^{k2-1}(\partial_j^{k2}\partial_i^{k1}\psi) (\varphi)$$

Two of these terms cancel and our total $\nabla\bullet\mathbf{J}$ result is then

$$\nabla\bullet\mathbf{J} \; = \; \psi(\partial_i^{k1} \partial_j^{k2}\varphi) - (-1)^{k1+k2} (\partial_j^{k2}\partial_i^{k1}\psi) \varphi$$

This is a *fairly impressive simplification* compared with our starting point which was

$$\nabla\bullet\mathbf{J} = \Sigma_{m=0}^{k1-1} (-1)^m [(\partial_i^{m+1}\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi) + (\partial_i^m\psi) (\partial_i^{k1-m} \partial_j^{k2}\varphi)]$$

$$+ (-1)^{k1} \Sigma_{m=0}^{k2-1} (-1)^m [(\partial_j^{m+1}\partial_i^{k1}\psi) (\partial_j^{k2-m-1} \varphi) + (\partial_j^m\partial_i^{k1}\psi) (\partial_j^{k2-m} \varphi)]$$

Now, as observed above, we can get $\nabla\bullet\mathbf{J}'$ by swapping k1↔k2 and i↔j in $\nabla\bullet\mathbf{J}$. Thus:

$$\nabla\bullet\mathbf{J} = \psi(\partial_i^{k1} \partial_j^{k2}\varphi) - (-1)^{k1+k2} (\partial_j^{k2}\partial_i^{k1}\psi) \varphi$$

$$\nabla\bullet\mathbf{J}' = \psi(\partial_j^{k2} \partial_i^{k1}\varphi) - (-1)^{k2+k1} (\partial_i^{k1}\partial_j^{k2}\psi) \varphi$$

But of course these are exactly the same, QED. We conclude that **J** and **J'** differ by a vector which has zero divergence! Here is that vector:

$$\Delta\mathbf{J} = \mathbf{J} - \mathbf{J}' =$$

$$\hat{\mathbf{i}} \quad \Sigma_{m=0}^{k1-1} (-1)^m (\partial_i^m\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi)$$

$$+ \hat{\mathbf{j}} \quad (-1)^{k1} \Sigma_{m=0}^{k2-1} (-1)^m (\partial_j^m\partial_i^{k1}\psi) (\partial_j^{k2-m-1} \varphi)$$

$$- \hat{\mathbf{j}} \quad \Sigma_{m=0}^{k2-1} (-1)^m (\partial_j^m\psi) (\partial_j^{k2-m-1}\partial_i^{k1}\varphi)$$

$$- \hat{\mathbf{i}} \quad (-1)^{k2} \Sigma_{m=0}^{k1-1} (-1)^m (\partial_i^m\partial_j^{k2}\psi) (\partial_i^{k1-m-1} \varphi)$$

$$= \hat{\mathbf{i}} \ \Sigma_{m=0}^{k1-1} (-1)^m \ \{ \ (\partial_i^m\psi) (\partial_i^{k1-m-1}\partial_j^{k2}\varphi) - (-1)^{k2}(\partial_i^m\partial_j^{k2}\psi) (\partial_i^{k1-m-1} \varphi) \ \}$$

$$+ \hat{\mathbf{j}} \ \Sigma_{m=0}^{k2-1} (-1)^m \ \{ \ (-1)^{k1}(\partial_j^m\partial_i^{k1}\psi) (\partial_j^{k2-m-1} \varphi) \ - (\partial_j^m\psi) (\partial_j^{k2-m-1}\partial_i^{k1}\varphi) \ \}$$

I think it is fair to claim that it is extremely non-obvious that this vector $\Delta\mathbf{J}$ as written here happens to have zero divergence! But this is exactly what we proved above. In general for more complex cases we shall soon encounter, one must keep in mind this non-obviousness.

4. Review of the conclusions of Section 3 above.

We found two ways to write our "double parts" integration theorem:

$$\int dV \ \psi(\partial_i^{k1}\partial_j^{k2}\varphi) = (-1)^{k1+k2} \int dV \ (\partial_j^{k2}\partial_i^{k1}\psi) \ \varphi + \int d\mathbf{S} \bullet \mathbf{J}$$

$$\int dV\ \psi(\partial_i^{k1}\partial_j^{k2}\varphi) = (-1)^{k1+k2} \int dV\ (\partial_j^{k2}\partial_i^{k1}\psi)\ \varphi + \int d\mathbf{S} \bullet \mathbf{J'}$$

where the two currents are given by

$$\mathbf{J} = \hat{\mathbf{i}} \sum_{m=0}^{k1-1} (-1)^m (\partial_i^m\psi)(\partial_i^{k1-m-1}\partial_j^{k2}\varphi) + \hat{\mathbf{j}}\ (-1)^{k1} \sum_{m=0}^{k2-1} (-1)^m (\partial_j^m\partial_i^{k1}\psi)(\partial_j^{k2-m-1}\varphi)$$

$$\mathbf{J'} = \hat{\mathbf{j}} \sum_{m=0}^{k2-1} (-1)^m (\partial_j^m\psi)(\partial_j^{k2-m-1}\partial_i^{k1}\varphi) + \hat{\mathbf{i}}\ (-1)^{k2} \sum_{m=0}^{k1-1} (-1)^m (\partial_i^m\partial_j^{k2}\psi)(\partial_i^{k1-m-1}\varphi)$$

The theorem is certainly true for either current, but the currents are different! The divergence of the difference is 0, which lets us understand, via the divergence theorem, why both currents work,

$$\int d\mathbf{S} \bullet \mathbf{J} = \int dV\ \nabla\bullet\mathbf{J} \qquad \int d\mathbf{S} \bullet \mathbf{J'} = \int dV\ \nabla\bullet\mathbf{J'} \qquad \nabla\bullet\mathbf{J} = \nabla\bullet\mathbf{J'}$$

Suppose we have a differential operator $L = \partial_i^{k1}\partial_j^{k2}$ and $L^* = (-1)^{k1+k2}\partial_i^{k1}\partial_j^{k2}$. The Green's Theorem (Stak 5.73) tells us that

$$\int (\psi\ L\varphi - \varphi\ L^*\psi) = \int d\mathbf{S} \bullet \mathbf{J} = \int d\mathbf{S} \bullet \mathbf{J'}$$

It is pretty clear that we could construct other viable currents in this manner:

$$\mathbf{J''} = \alpha\ \mathbf{J} + (1-\alpha)\mathbf{J'}$$

It may turn out for certain operators L that one form of the current is more "symmetrical" and simple than some other form of the current.

5. Parts Integration for a general differential operator

Stakgold likes to write his general differential operator of order p as follows:

$$L = \sum_{|k|\le p}\ a_k D^k = \sum_{|k|\le p}\ a_{k1k2\ldots kn}\ \partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3}\ \ldots\ \partial_n^{kn}$$

where $|k| = \sum_{i=1}^{m} k_i$ and where in general the $a_{k1k2\ldots kn}$ are functions of $\mathbf{r}$ in R_n. The formal adjoint of L acting on a function u is given by

$$L^*u = \sum_{|k|\le p}\ (-1)^{|k|}\ \partial_1^{k1}\partial_2^{k2}\ \partial_3^{k3}\ \ldots\ \partial_n^{kn}(a_{k1k2\ldots kn}\ u)$$

This form arises basically from the fancy parts integration we are about to do. We are interested in the current **J** which appears in this general Green's Theorem in R_n space:

$$\int dV\ (vLu - uL^*v) \;=\; \int d\mathbf{S}\bullet\mathbf{J} \;=\; \int dV\ \nabla\bullet\mathbf{J} \qquad\qquad vLu - uL^*v = \nabla\bullet\mathbf{J}$$

Our first task will be to show that there in fact exists such a current. Once we show this, we know from previous sections that the current **J** won't be unique and any other **J'** which has the same divergence as **J** will be just as valid, and we suspect that some forms of **J** will be "nicer" than others.

It suffices to consider a single term in the sum that comprises L. We will show that a current **J** exists for such a term, and then we can add the currents for the various terms in L to get a total current. This last step is not very significant since everything is linear.

Our starting position is then this

$$\int dV\ (vLu - uL^*v) =$$

$$\int dV\ [\ \ (v\ a_{k1k2\ldots kn}\ \partial_1{}^{k1}\partial_2{}^{k2}\ \partial_3{}^{k3}\ \ldots.\ \partial_n{}^{kn}\ u$$

$$-\ u\ (-1)^{k1+k2\ldots+kn}\ \partial_1{}^{k1}\partial_2{}^{k2}\ \partial_3{}^{k3}\ \ldots.\ \partial_n{}^{kn}\ (a_{k1k2\ldots kn}\ v)\ ]$$

The general idea is that, in the second term, we "throw over" (from right to left) the derivatives one at a time. Although we could do this in any order we want, I will throw over $\partial_1{}^{k1}$ first, then $\partial_2{}^{k2}$ second, and so on in that order. It is no surprise to realize that for each "ordering" one chooses for doing this, one gets a different current! One could do $\partial_3{}^{k3}$ first, then $\partial_1{}^{k1}$ and proceed in some strange order. Or one could be really obstinate and throw over $\partial_3{}^{k3-1}$ first, then $\partial_1{}^{k1}$, then $\partial_3{}^{1}$, and so on. There are of course a factorially large number of ways to do this, and I am picking one way which seems pretty reasonable.

When we throw over $\partial_1{}^{k1}$ we pick up a "surface term", and we are one step closer with our non-surface term to the expression vLu which is the first term above (and which we are eventually going to cancel out). Since we are throwing over a derivative $\partial_1{}^{k1}$ with respect to coordinate x_1, the surface term has the form $\int d\mathbf{S}\bullet\hat{\mathbf{1}}\ (J_1)$ so that the current for this term is simply $\hat{\mathbf{1}}\ (J_1)$. This follows from the Theorem of Section 2 above,

$$\int dV\ \psi(\partial_i{}^k\varphi) \;= (-1)^k \int dV\ (\partial_i{}^k\psi)\ \varphi \;+\; \int d\mathbf{S}\bullet\ \mathbf{J} \ \text{ where } \ \mathbf{J} = \hat{\mathbf{i}}\ \ \{\ \Sigma_{m=0}{}^{k-1}\ (-1)^m\ (\partial_i{}^m\psi)\ (\partial_i{}^{k-m-1}\varphi)\ \ \}$$

where we set i = 1 and k = k1. Next, we throw over the $\partial_2{}^{k2}$ derivative and that creates a surface term which has the form $\int d\mathbf{S}\bullet\hat{\mathbf{2}}\ (J_2)$. We keep doing this until we have thrown over $\partial_n{}^{kn}$ and then we are done. The residual integral cancels the first vLu term, and we have nothing left but our n surface integrals. Here are the expressions for the first three currents that we develop in this process: (see raw notes for details)

$$J_1 = -\ (-1)^{k1+k2\ldots+kn}\ \Sigma_{m=0}{}^{k1-1}(-1)^m\ (\partial_1{}^m \qquad u)\ \ (\partial_1{}^{k1-m-1}\ [\partial_2{}^{k2}\ \partial_3{}^{k3}\ \ldots.\ \ \partial_n{}^{kn}\ (a_{k1k2\ldots kn}\ v)\ ])$$

$$J_2 = -\ (-1)^{k2\ldots+kn}\ \ \ \Sigma_{m=0}{}^{k2-1}(-1)^m\ (\partial_2{}^m \qquad \partial_1{}^{k1}\ u)\ (\partial_2{}^{k2-m-1}\ [\qquad \partial_3{}^{k3}\partial_4{}^{k4}.\ \ \partial_n{}^{kn}\ (a_{k1k2\ldots kn}\ v)\ ])$$

$$J_3 = -\ (-1)^{k3\ldots+kn}\ \ \ \Sigma_{m=0}{}^{k3-1}\ (-1)^m\ \ (\partial_3{}^m \ \ \partial_2{}^{k2}\partial_1{}^{k1}\ u)\ (\partial_3{}^{k3-m-1}\ [\qquad\quad \partial_4{}^{k4}\ \ldots.\ \partial_n{}^{kn}\ (a_{k1k2\ldots kn}\ v)\ ])$$

You can see that derivatives in the currents are piling up on u and moving away from $(a_{k1k2\ldots kn}\, v)$. These currents are really quite obvious, we just keep using our theorem above with different values of these objects: k, i, ψ and φ. We can write an expression for the sth surface current as follows:

$$J_s = -(-1)^{ks\ldots+kn} \sum_{m=0}^{ks-1} (-1)^m \left(\partial_s^{\,m} \left[\partial_{s-1}^{\,k(s-1)} \ldots \partial_2^{\,k2} \partial_1^{\,k1}\, u\right]\right) \left(\partial_s^{\,ks-m-1} \left[\partial_{s+1}^{\,k(s+1)} \ldots \partial_n^{\,kn} \left(a_{k1k2\ldots kn}\, v\right)\right]\right)$$

(up over the sign; down over the first sum; up over the second bracket)

Here the words up and down mean that numbers in that area are incrementing up or down. If this idea is unclear, just look at the first few currents shown above.

Our Green's Theorem is then this (we have just proven it!):

$$\int dV\, (vLu - uL^*v) = \int d\mathbf{S}\bullet\mathbf{J} \qquad \text{where} \quad \mathbf{J} = \sum_{s=1}^{n} \hat{\mathbf{s}}\, J_s$$

with the J_s as just stated above. One realizes that $\partial^0 = 1$ and that impossible negative summation terms are of course 0. So, by simply applying our Part 2 Theorem n times, we have proven conclusively that it is possible to find a current **J** so that the above is true. We have explicitly constructed such a current, and as stated, we know it is not unique. From the divergence theorem, we also have shown that

$$(vLu - uL^*v) = \nabla\bullet\mathbf{J}$$

No matter how you "move the derivatives", the resulting current **J** is going to be "bilinear" in u and v, which just means that in each current component, u appears once and v appears once, each being acted upon by some differential operator. This bilinear fact again goes back to our Theorem which is bilinear in ψ and φ, no matter at what "throw over" stage you apply the theorem.

Thus, we have proven the claims that Stakgold makes on the top of page 40 of Volume 2, Chapter 5. When I first saw these claims, I was truly mystified, but now it is old hat. The only tools needed for the derivation are the R_n versions of the famous divergence theorem (1) and the less well-known "volume integral of a gradient theorem" (2) on my page of integral theorems (which is really just a special case of the divergence theorem.)

Example 1: The Laplacian

Let $L = \nabla^2$ so that $a_k = 1$. The term $\partial_i^{\,2}$ has $k_i = 2$ and all other $k_j = 0$. This means that all the current components above except J_i vanish (due to the m sums staring at m = -1 !), and we have

$$J_i = -(-1)^2 \sum_{m=0}^{1} (-1)^m (\partial_i^{\,m}\, u)\ \partial_i^{\,1-m}\, v$$

$$= -\{\, u\, \partial_i v - (\partial_i\, u)\, v \,\} = v(\partial_i\, u) - u\,(\partial_i v)$$

so that $\mathbf{J} = v\,\nabla u - u\nabla v$ just as is claimed in Stak 5.74 (and as derived above in one of our first examples). There are many forms for this current, but this is certainly a simple form!

Example 2: The Biharmonic ∇^4 operator $\qquad \nabla^4 = \nabla^2\nabla^2 = \Sigma_{ij}\, \partial_i^2 \partial_j^2$

This is the example that set me off on a multi-day bout of confusion that is now resolved by the various comments given above. The current you obtain always depends on the order in which you throw things over, and some orderings give not very simple results!

A very "symmetric" approach is to think of the problem in this way

$$\int dV\ (vLu - uL^*v) \;=\; \Sigma_{ij} \int dV\ \{\, v\,(\partial_i^2 \partial_j^2 u) \;- u\,(\partial_i^2 \partial_j^2 v)\,\}$$

where we *ignore* the fact that off-diagonal terms are like $\partial_1^2 \partial_2^2$ and diagonal terms are like ∂_2^4 . We simply "throw over" ∂_i^2 and then we throw over ∂_j^2 in the manner described above. Notice that, for diagonal terms like ∂_2^4, this means we throw over ∂_2^2 and then we throw over another ∂_2^2, but these two processes I think give the exact same result current wise. I do all this in the raw notes, and the current you get is this:

$$-\mathbf{J} = \hat{\mathbf{i}}\ \{\, \Sigma_{m=0}^{1} (-1)^m\, (\partial_i^m u)\, (\partial_i^{1-m} (\partial_j^2 v))\ \} + \hat{\mathbf{j}}\ \{\, \Sigma_{m=0}^{1} (-1)^m\, (\partial_j^m (\partial_i^2 u))\, (\partial_j^{1-m} v)\ \}$$

$$= \hat{\mathbf{i}}\ \{\, u\,(\partial_i\, \partial_j^2 v) \;- (\partial_i u)\, (\partial_j^2 v)\,\} + \hat{\mathbf{j}}\ \{\, (\partial_i^2 u)\, (\partial_j\, v) \;- (\partial_j\, \partial_i^2 u)\,\}$$

We can now apply our summation Σ_{ij} separately to the two terms of the current. In each term *one* of the sums can be moved forward to create a ∇^2 object, and we get

$$-\Sigma_{ij}\, \mathbf{J} = \Sigma_i\ [\quad \hat{\mathbf{i}}\ \{\, u\,(\partial_i \nabla^2 v) \;- (\partial_i u)\, (\nabla^2 v)\,\}\ ] + \Sigma_j\ [\ \hat{\mathbf{j}}\ \{(\nabla^2 u)(\, \partial_j v) - (\partial_j \nabla^2 u) v\ \}\quad]$$

Now in the second term, replace the dummy change the dummy index from j to i

$$-\mathbf{J} = \Sigma_i\ [\quad \hat{\mathbf{i}}\ \{\, u\,(\partial_i \nabla^2 v) \;- (\partial_i u)\, (\nabla^2 v)\,\}\ ] + \Sigma_i\ [\ \hat{\mathbf{i}}\ \{(\nabla^2 u)(\, \partial_i v) - (\partial_i \nabla^2 u) v\ \}\quad]$$

Now combine the two sums

$$-\mathbf{J} = \Sigma_i\, \hat{\mathbf{i}}\ \{\, u\,(\partial_i \nabla^2 v) \;- (\partial_i u)\, (\nabla^2 v) \;+ (\nabla^2 u)(\, \partial_i v) - (\partial_i \nabla^2 u) v\ \}$$

Then conclude that

$$-J_i = u\,(\partial_i \nabla^2 v) \;- (\partial_i u)\, (\nabla^2 v) \;+ (\nabla^2 u)(\, \partial_i v) - (\partial_i \nabla^2 u) v$$

which means

$$-\mathbf{J} = u\,(\mathbf{grad}\ \nabla^2 v) \;- (\mathbf{grad}\ u)\, (\nabla^2 v) \;+ (\nabla^2 u)(\, \mathbf{grad}\ v) - (\mathbf{grad}\ \nabla^2 u) v$$

which agrees exactly with Stak page 40. So this is the right way to derive a simple looking current.

But what happens if we apply our "general formula"? In that formula, we are supposed to rigorously treat each term in the ij sum as having the official ordered form we have described. We can write out these terms as follows

$$L = \Sigma_{i=1}^{n} \partial_i^4 + 2 \Sigma_{i>j} \partial_i^2 \partial_j^2$$

We then "process" each term according to our general formula to get the current for that term. Here are some of the ingredients (see raw notes for full painful detail)

For term $\partial_i^2 \partial_j^2$ with i > j we get only two current contributions:

$$J_i = -(-1)^2 \Sigma_{m=0}^{1} (-1)^m \left(\partial_i^m [\partial_j^2 u]\right) \left(\partial_i^{1-m} [(v)]\right)$$

$$= - \{(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v\}$$

$$J_j = -(-1)^{2+2} \Sigma_{m=0}^{1} (-1)^m (\partial_j^m u)(\partial_j^{1-m} \partial_i^2 v)$$

$$= - \{ u (\partial_j \partial_i^2 v) - (\partial_j u)(\partial_i^2 v)$$

For term ∂_i^4 we get just one current contribution:

$$J_i = -(-1)^4 \Sigma_{m=0}^{3} (-1)^m (\partial_i^m u)(\partial_i^{3-m} v)$$

$$= - \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

If we then add the currents of our terms together and fiddle around quite a bit, we can get the current to be in this form:

$$-J_i = \{ u (\partial_i^3 v) - (\partial_i u)(\partial_i^2 v) + (\partial_i^2 u)(\partial_i v) - (\partial_i^3 u) v \}$$

$$+ 2\Sigma_{j=1}^{i-1} [(\partial_j^2 u)(\partial_i v) - (\partial_i \partial_j^2 u) v]$$

$$+ 2\Sigma_{j=i+1}^{n} [u (\partial_i \partial_j^2 v) - (\partial_i u)(\partial_j^2 v)]$$

This result is awesomely ugly in that it includes two different summations

1 2 3 ..... i-1 i i+1 ..... n
<---- first sum ---> <---second sum --->

which tend to form only "partial" Laplacians. The ugliness arises from the fact that, using our general formula, we are throwing things over in a strange way. For example, we have $\partial_i^2 \partial_j^2$ treated only for i>j which means we treat this term always as $\partial_j^2 \partial_i^2$ for its throwing order. Thus, in effect, both the terms $\partial_1^2 \partial_2^2$ and $\partial_2^2 \partial_1^2$ are "thrown over" in the same manner, whereas in our first solution above, we throw things over in the order in which they appear. In other words, for the term $\partial_2^2 \partial_1^2$, we would be throwing

over ∂_2^2 first in our symmetric solution, but we are throwing over ∂_1^2 first in our "systematic" general solution. This bad batting order results in the very messy (but still correct I am quite sure) current shown above. We know that my messy current and Stakgold's simple current must differ by a vector which has 0 divergence. The simple current was

$$-J_i' = u\,(\partial_i \nabla^2 v) - (\partial_i u)\,(\nabla^2 v) + (\nabla^2 u)(\partial_i v) - (\partial_i \nabla^2 u)v$$

Somehow this simple J_i' and my messy J_i must have the same divergence, but you can see that even for the simple current, applying ∂_i creates 8 messy terms. Somehow application of ∂_i to J_i will make those partial sums go away. Terms will somehow cancel, and the sums will combine with isolated terms to form full sums which will then yield full ∇^2 objects.

By the way, I get the exact same messy result if I start this more symmetric way:

$$L = \Sigma_{i=1}^{n}\, \partial_i^4 + \Sigma_{i>j}\, \partial_i^2 \partial_j^2 + \Sigma_{j>i}\, \partial_i^2 \partial_j^2$$

and I compute the currents for the *three* types of terms.

So one conclusion is this: my general systematic formula for the surface current is useful inasmuch as it provides by construction an explicit current which proves Green's Theorem. But when it comes time to compute a useful current for a specific operator L, it is best to find a "symmetric" looking method for "throwing things over" that will give a simple looking current. This is a general concept that one sees many times in physics, such as in Goldstein mechanics.