Finding Green's Functions by the Dirichlet Method PhL 4.23.10

Stak chap 6 has various Green's-finding methods and probably everything is there. Here I just want to ponder a few situations to seek better comprehension.

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Overview.

In **Problem 1**, I just review a very simple Dirichlet problem having nothing to do with any Green's functions -- it is just Stak's little opening gambit, the 2D unit circle. The general Dirichlet method is to first write a Smythian form where you arrange that the Dirichlet prescribed boundary aligns with a constant value of a non-oscillatory coordinate of your atomic form. Then you solve for the coefficient(s) by using orthogonality in the other oscillatory dimensions. You then have the coefficients, so you can then stick them into the Smythian form and you are done. Perhaps you can do some sums or integrals in this form to simplify the result.

In **Problem 2**, I discuss the Dirichlet problem in 3D where the bounding surface is a Great Hemisphere and where V is prescribed on the flat equatorial disk of this hemisphere and is 0 on the curved part. I decided to use Cartesian coordinates for this problem, since I thought I could make the prescription surface line up with coordinate = constant. The first step is to sort out the Cartesian atomic forms and throw out those that don't fit the problem. I then arrive at this Smythian form for such a problem

$$V(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; A(k_x, k_y) \exp(ik_x x) \exp(ik_y y) \exp(-\sqrt{k_x^2+k_y^2}\, z)$$

where the continuous spectrum arises from the infinite size of the closed surface. That is, there is nothing to cause the k_x and k_y values to be quantized. I took the hemisphere to be $z \geq 0$ and the prescription plane to be $z = 0$. You see how the plane aligns with a constant value of the non-oscillatory coordinate z, and of course we are decaying in z. The x and y coordinates are oscillatory. The "Laplace condition" is $k_z^2 = k_x^2 + k_y^2$ for the way I have set things up. If we require the potential to match the value f(x,y) on the z=0 plane, we get

$$f(x,y) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; A(k_x, k_y) \exp(ik_x x) \exp(ik_y y)$$

and this is simply a double Fourier Transform, so the coefficients are given by:

$$A(k_x, k_y) = (1/2\pi)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(-ik_x x) \exp(-ik_y y) f(x,y)$$

All the details are given below.

In **Problem 3** I demonstrate what I call "the Dirichlet method" of finding a Green's Function for the closed surface described in Problem 2. I first put a point charge +q at location (0,0,a) and I compute the negative of the potential this point charge creates on the z = 0 math plane, namely.

$$f(x,y) = -q / \sqrt{x^2 + y^2 + a^2}$$

Since this function is even in x and y, it turns out that we want Fourier Cosine Transforms, because such things are not replicated in the exponential table. In other words, our coefficient formula is this:

$$A(k_x, k_y) = 2^2 (1/2\pi)^2 \int_{0}^{\infty} dx \int_{0}^{\infty} dy \cos(k_x x) \cos(k_y y) \{-q / \sqrt{x^2 + y^2 + a^2} \}$$

I do the integrals one at a time, and find everything I need in the Bateman Vol 4 tables. To reduce symbols, I use $r = k_x$ and $s = k_y$. The first integral transform results in a certain K_o function and we get this intermediate result:

$$A(k_x, k_y) = 2^2 (-q) (1/2\pi)^2 \int_{0}^{\infty} dx \cos(k_x x) K_0(\sqrt{x^2 + a^2}\, k_y)$$

Then the second integral takes us back to elementary functions and we find:

$$A(k_x, k_y) = (-q/2\pi) \exp(- a\sqrt{k_x^2 + k_y^2}\,) / \sqrt{k_x^2 + k_y^2}$$

So we started with a 1/R elementary function $1/\sqrt{x^2 + y^2 + a^2}$, moved to $K_0(\sqrt{x^2 + a^2}\, k_y)$, and came back to elementary e^{-aR}/R where the new R is in k-space.

I then insert this coefficient into the Smythian form shown above and which I repeat here:

$$V(x,y,z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; A(k_x, k_y) \exp(ik_x x) \exp(ik_y y) \exp(-\sqrt{k_x^2 + k_y^2}\, z)$$

Again, the coefficient is even in its two arguments so we really have this:

$$V(x,y,z) = 2^2 \int_{0}^{\infty} dk_x \int_{0}^{\infty} dk_y \; A(k_x, k_y) \cos(k_x x) \cos(k_y y) \; \exp(-\sqrt{k_x^2 + k_y^2}\, z)$$

Once again we face a series of two Cosine transforms. We start with this pure exponential and after doing the y transform we have a K_o function of a similar argument form:

$$V(x,y,z) = 2^2(-q/2\pi) \int_{0}^{\infty} dr \cos(k_x x) K_0(k_x \sqrt{(z+a)^2 + y^2}\,)$$

Then this integral returns us to elementary functions and we have

$$V(x,y,z) = \; - q \,/\, \sqrt{x^2 + y^2 + (z+a)^2}$$

which we (thankfully) recognize as the potential of the image charge at location (0,0,-a). We then add our resulting potential V(x,y,z) to our original point charge potential +q/R and the result is the famous Green's function for a point charge near a grounded plane. The image method is of course much easier, but I wanted to grind this through to demonstrate how the "Dirichlet method of finding a Green's function" works in what must be about the simplest case one could imagine.

In **Problem 4** I again demonstrate "the Dirichlet method" of finding a Green's Function for the closed surface being a sphere. This time I use spherical coordinates. The prescription potential on the sphere here is given by

$$V(b,\theta,\varphi) = -q/\sqrt{a^2 + b^2 - 2ab\cos\theta} \;\; = \; f(\theta,\varphi) \qquad\qquad = -q/R$$

The atomic form for the potential which we want to match this boundary condition is

$$V(r,\theta,\varphi) \;= \Sigma_n \, A_n \, (r/b)^n \;\; P_n(\cos\theta)$$

and without doing a single integral, just using the fact that

$$1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} \;\; = \Sigma_{n=0}^{\infty} \;\; P_n(\cos\theta)\; \alpha^n \qquad\qquad \alpha < 1,$$

I am able to show that

$$V(r,\theta,\varphi) \;= -q\,(b/a) \,/\, \sqrt{d^2+r^2-2rd\cos(\theta)}$$

where $d = b^2/a$ is the distance up the z axis of an "image charge" whose size is -q (b/a) . So once again, we have derived the famous image charge result for the Green's function of a sphere.

Problem 1: A Review of How Stak does Dirichlet for the 2D unit circle.

The first step was to write the potential in a Smythian form of atomic terms, which for that problem in 2D was this, where we use "2D polar coordinates",

$$V(r,\theta) = \Sigma_n \, a_n r^{|n|} e^{in\theta}$$

Then on the unit circle we have

$$V(1,\theta) = \Sigma_n \, a_n e^{in\theta} \;\; = f(\theta), \qquad\qquad f(\varphi) = \text{a prescribed potential on the surface}$$

Recently in my diagonalization document in math/group theory, I talked about the complex Fourier Series transform in the context of group G = SO(2), and here was the result

$f^n = (1/2\pi) \int d\theta\, f(\theta)\, e^{+in\theta}$ // projection //Complex Fourier Series Transform

$f(\theta) = \Sigma_n f^n e^{-in\theta}$ // expansion

$(1/2\pi) \int d\theta\, e^{in\theta} e^{-in'\theta} = \delta_{nn'}$ // orthogonality

$(1/2\pi) \Sigma_n e^{-in\theta'} e^{in\theta} = \delta(\theta-\theta')$ // completeness

So we take our Smythe form and apply $(1/2\pi) \int d\theta\, e^{-in'\theta}$ to both sides:

$\Sigma_n a_n e^{in\theta} = f(\theta)$

$\Sigma_n a_n (1/2\pi) \int d\theta\, e^{-in'\theta} e^{in\theta} = (1/2\pi) \int d\theta\, e^{-in'\theta} f(\theta)$

$\Sigma_n a_n \delta_{n,n'} = (1/2\pi) \int d\theta\, e^{-in'\theta} f(\theta) = a_{n'}$ // using orthogonality above

$a_n = (1/2\pi) \int d\theta\, e^{-in\theta} f(\theta)$

The problem is now completely solved:

$V(r,\theta) = \Sigma_n a_n r^{|n|} e^{in\theta}$ where $a_n = (1/2\pi) \int d\theta\, e^{-in\theta} f(\theta)$

Stak then went an extra step and replaced the sum on n with an integral to get his form 6.11 with its famous Poisson Kernel integrated against f(θ). But we don't need that for our present discussion. The point is that the way we know how to solve a Dirichlet problem is to assume a Smythian form expanded on a set of complete functions with some coefficients, then we use the orthogonality of those atomic functions to compute the coefficients and then we are done.

Problem 2: Dirichlet problem for "the great hemisphere" with V prescribed on the planar boundary.

Imagine a plane on which is prescribed some arbitrary potential V(**s**), but which potential vanishes at all far reaches of the plane. Add to this planar surface a "great sphere" surface on which V = 0. We then have a closed boundary (a "great hemisphere") on which the potential is prescribed everywhere. We want to know the potential everywhere inside the boundary. This is a Dirichlet problem in 3D. For the geometry just mentioned, how do we solve this problem?

The first big problem now is selecting coordinates. We would like to have our "plane" correspond to one of the three coordinates being a constant, and we would like that coordinate to be "the non-oscillatory" one so we can use orthogonality on the other two dimensions. This is the way it works for a spherical boundary type problem.

Let's just try this in Cartesian coordinates. The atoms are discussed in "why spherical atoms don't work", where we find a most general atomic form that looks like this:

$$\varphi_{\mathbf{k}}(x,y,z) = [A\exp(k_{\mathbf{x}}x)+B\exp(-k_{\mathbf{x}}x)+Cx+D] * [A'\exp(k_{\mathbf{Y}}y)+B'\exp(-k_{\mathbf{y}}y)+C'y+D'] * [A''\exp(k_{\mathbf{z}}z)+B''\exp(-k_{\mathbf{z}}z)+C''z+D''] \quad \text{where } k_{\mathbf{z}}^2 = -k_{\mathbf{x}}^2 - k_{\mathbf{y}}^2$$

In our plane problem with V = 0 far away, I am guessing that the homogeneous things like Cx+D won't be present, so we might then have

$$V(x,y,z) = \int dk_{\mathbf{x}}dk_{\mathbf{y}}dk_{\mathbf{z}} \qquad \text{// subject to } k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2 = k_{\mathbf{z}}^2$$

$$[A_{\mathbf{kx}}\exp(ik_{\mathbf{x}}x)+ B_{\mathbf{kx}}\exp(-ik_{\mathbf{x}}x)]\ [A_{\mathbf{ky}}\exp(ik_{\mathbf{Y}}y)+ B_{\mathbf{ky}}\exp(-ik_{\mathbf{y}}y)]\ [A_{\mathbf{kz}}\exp(k_{\mathbf{z}}z)+ B_{\mathbf{kz}}\exp(-k_{\mathbf{z}}z)]$$

Here I have made x and y be oscillatory, and z be expo. Let's have our hemi-great sphere be $z \geq 0$, then we should be able to further restrict this way

$$V(x,y,z) = \int dk_{\mathbf{x}}dk_{\mathbf{y}}dk_{\mathbf{z}} \qquad \text{// subject to } k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2 = k_{\mathbf{z}}^2$$

$$[A_{\mathbf{kx}}\exp(ik_{\mathbf{x}}x)+ B_{\mathbf{kx}}\exp(-ik_{\mathbf{x}}x)]\ [A_{\mathbf{ky}}\exp(ik_{\mathbf{Y}}y)+ B_{\mathbf{ky}}\exp(-ik_{\mathbf{y}}y)]\ \exp(-k_{\mathbf{z}}z)]$$

Now let's say the integrals run $-\infty$ to ∞ so we then have, where $k_{\mathbf{x}}^2 + k_{\mathbf{y}}^2 = k_{\mathbf{z}}^2$,

$$V(x,y,z) = \int dk_{\mathbf{x}}dk_{\mathbf{y}}dk_{\mathbf{z}}\ A(k_{\mathbf{x}}, k_{\mathbf{y}}, k_{\mathbf{z}})\exp(ik_{\mathbf{x}}x)\exp(ik_{\mathbf{Y}}y)\exp(-k_{\mathbf{z}}z)$$

Maybe write one more time:

$$V(x,y,z) = \int dk_{\mathbf{x}}dk_{\mathbf{y}}\ A(k_{\mathbf{x}}, k_{\mathbf{y}})\exp(ik_{\mathbf{x}}x)\exp(ik_{\mathbf{Y}}y)\exp(-\sqrt{k_{\mathbf{x}}^2+k_{\mathbf{y}}^2}\ z)$$

If we now apply our prescribed potential on the z = 0 plane we get

$$V(x,y,0) = \int dk_{\mathbf{x}}\ dk_{\mathbf{y}}\ A(k_{\mathbf{x}}, k_{\mathbf{y}})\exp(ik_{\mathbf{x}}x)\exp(ik_{\mathbf{Y}}y) = f(x,y)$$

Then we should be able to do a double Fourier Integral transform to get our coefficients. This transform is given by the following, in one variable,

$$F(k) = (1/2\pi)\int dx\ f(x)\ e^{+ikx} \qquad \text{// projection}$$

$$f(x) = \int dk\ F(k)\ e^{-ikx} \qquad \text{// expansion}$$

$$\int dx/2\pi\ e^{ikx}\ e^{-ik'x} = \delta(k-k') \qquad \text{// orthogonality}$$

$$\int dk/2\pi\, e^{-ikx'}\, e^{ikx} = \delta(x'-x) \qquad \text{// completeness}$$

So, apply $(1/2\pi)^2 \int dx\, dy \exp(-ik_x'x) \exp(-ik_y'y)$ to both sides to get

$$\int dk_x\, dk_y\, A(k_x, k_y) \exp(ik_x x) \exp(ik_y y) = f(x,y)$$

$$(1/2\pi)^2 \int dk_x\, dk_y\, A(k_x, k_y) \int dx\, dy \exp(-ik_x'x) \exp(-ik_y'y) \exp(ik_x x) \exp(ik_y y)$$

$$= (1/2\pi)^2 \int dx\, dy \exp(-ik_x'x) \exp(-ik_y'y) f(x,y)$$

$$\int dk_x\, dk_y\, A(k_x, k_y) \int dx/2\pi\, dy/2\pi \exp(-ik_x'x) \exp(-ik_y'y) \exp(ik_x x) \exp(ik_y y)$$

$$= (1/2\pi)^2 \int dx\, dy \exp(-ik_x'x) \exp(-ik_y'y) f(x,y)$$

$$\int dk_x\, dk_y\, A(k_x, k_y)\, \delta(k_x - k_x')\, \delta(k_y - k_y') = A(k_x', k_y')$$

$$= (1/2\pi)^2 \int dx\, dy \exp(-ik_x'x) \exp(-ik_y'y) f(x,y)$$

$$A(k_x, k_y) = (1/2\pi)^2 \int dx\, dy \exp(-ik_x x) \exp(-ik_y y) f(x,y)$$

So we think then that we have solved this Dirichlet problem. The solution is this:

$$V(x,y,z) = \int dk_x dk_y\, A(k_x, k_y) \exp(ik_x x) \exp(ik_y y) \exp(-\sqrt{k_x^2 + k_y^2}\, z)$$

$$\text{where } A(k_x, k_y) = (1/2\pi)^2 \int dx\, dy \exp(-ik_x x) \exp(-ik_y y) f(x,y)$$

If everything converges, we have a solution to the Laplace equation which meets our boundary condition prescribed on the z = 0 plane and hopefully is OK at ∞. Since this solution is unique, this is it.

Problem 3: Point charge and a Plane by Dirichlet Method

We have a +q point charge on one side of a metal plane held at V = 0, all in 3D. The potential everywhere is known as the Green's Function. The point charge is the "excitation", like the point pressure on a string in 1D. It is possible to think of the Green's Function as the solution to a Dirichlet (no charges) problem in the following way. First, take the +q point charge and take note of the potential it causes on the math plane located where our metal plane will go. Schematically, that potential is V = q/R where R is the distance from that +q point charge to a point on the plane. Second, get rid of the +q charge, but retain and negate that potential on the math plane and treat it as a Dirichlet "prescribed potential on a surface". Then

solve the Laplace equation inside this surface for this Dirichlet problem. Then when you add back the +q point charge, the total potential is the Green's Function because it solves Laplace and V = 0 on the plane.

Let's try it. We put our initial +q charge at location z = +a on the z axis. What is the potential on the z = 0 plane? R = distance from point (x,y,0) on plane to our +q charge, so $R^2 = x^2 + y^2 + a^2$. We are then supposed to take our Dirichlet potential to be -q/R, so we then have this prescription:

$$f(x,y) = -q / \sqrt{x^2 + y^2 + a^2}$$

We then want to compute our coefficients:

$$A(k_x, k_y) = (-q)\,(1/2\pi)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(-ik_x x) \exp(-ik_y y) / \sqrt{x^2 + y^2 + a^2}$$

Maybe save some symbols writing this as

$$A(r, s) = (-q)\,(1/2\pi)^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp(-irx) \exp(-isy) / \sqrt{x^2 + y^2 + a^2}$$

I could have used cylindrical coordinates and it is easy to write this double integral using ρ and θ, but for now leave as is. Let's try brute force. The first integral we need is

$$\int_{-\infty}^{\infty} dy \exp(-isy) / \sqrt{x^2 + y^2 + a^2} = \int_{-\infty}^{\infty} dy \exp(-isy) / \sqrt{y^2 + c^2} \qquad c^2 = x^2 + a^2$$

This is just a Fourier Transform and we should be able to look it up in the third section of Bateman volume 4 (hurray!). Well, this one does not appear as I would like, so try GR instead. I guess we can just keep the cosine piece and look for

$$= \int_{-\infty}^{\infty} dy \cos(sy) / \sqrt{y^2 + c^2} = 2 \int_{0}^{\infty} dy \cos(sy) / \sqrt{y^2 + c^2}$$

So this appears in the Fourier Cosine Transform section where I now quote: (p 9)

	$f(x)$	$g(y) = \int_0^\infty f(x) \cos(xy)\,dx \qquad y > 0$
(17)	$(x^2 + a^2)^{-\frac{1}{2}}$ Re $a > 0$	$K_0(ay)$

So we then know that

$$\int_{0}^{\infty} dy \cos(sy) / \sqrt{y^2 + c^2} = K_0(cs) = K_0(\sqrt{x^2 + a^2}\, s)$$

We then have this intermediate result:

$$A(r, s) = 2\,(-q)\,(1/2\pi)^2 \int_{-\infty}^{\infty} dx\, \exp(-irx)\, K_0(\sqrt{x^2 + a^2}\, s)$$

$$= 2^2\,(-q)\,(1/2\pi)^2 \int_{0}^{\infty} dx\, \cos(rx)\, K_0(\sqrt{x^2 + a^2}\, s)$$

I am amazed to find this transform in the same Bateman volume 4 table, quoting again: (p 56)

(43)	$K_0[a(x^2+\beta^2)^{\frac{1}{2}}]$ $\mathrm{Re}\ a > 0, \quad \mathrm{Re}\ \beta > 0$	$(\pi/2)(y^2+a^2)^{-\frac{1}{2}} e^{-\beta(y^2+a^2)^{\frac{1}{2}}}$

So we then know that

$$\int_{0}^{\infty} dx\, \cos(rx)\, K_0(\sqrt{x^2 + a^2}\, s) \;=\; (\pi/2)\, \exp(-a\sqrt{r^2+ s^2})/\sqrt{r^2+ s^2}$$

So we then have this final result for our coefficients:

$$A(r, s) = 2^2\,(-q)\,(1/2\pi)^2\,(\pi/2)\, \exp(-a\sqrt{r^2+ s^2})/\sqrt{r^2+ s^2} \;=\; (-q/2\pi)\, \exp(-a\sqrt{r^2+ s^2})/\sqrt{r^2+ s^2}$$

which we translate back to

$$A(k_x, k_y) = (-q/2\pi)\, \exp(-a\sqrt{k_x^2+ k_y^2})/\sqrt{k_x^2+ k_y^2}$$

Our Dirichlet problem solution should then be

$$V(x,y,z) = \int dk_x dk_y\; A(k_x, k_y)\, \exp(ik_x x)\, \exp(ik_y y)\, \exp(-\sqrt{k_x^2+k_y^2}\, z)$$

$$= (-q/2\pi) \int dk_x dk_y\, \exp(ik_x x)\, \exp(ik_y y)\, \exp[-\sqrt{k_x^2+k_y^2}\,(z+a)]/\sqrt{k_x^2+ k_y^2}$$

Now we have more integrals to do! Let's go back to our r,s notation

$$= (-q/2\pi) \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} ds\, \exp(irx)\, \exp(isy)\, \exp[-\sqrt{r^2+s^2}\,(z+a)] / \sqrt{r^2+ s^2}$$

We can see that the integrand is even in both r and s (apart from expos) so this is going to be

$$= \; 2^2(-q/2\pi) \int_0^\infty dr \int_0^\infty ds \cos(rx) \cos(sy) \exp[-\sqrt{r^2+s^2}\,(z+a)] / \sqrt{r^2+s^2}$$

$$= 2^2(-q/2\pi) \int_0^\infty dr \cos(rx) \int_0^\infty ds \cos(sy) \{ \exp[-\sqrt{r^2+s^2}\,(z+a)] / \sqrt{r^2+s^2} \}$$

So now we want to find this somewhere in Bateman

$$\int_0^\infty ds \cos(sy) \{ \exp[-\sqrt{r^2+s^2}\,(z+a)] / \sqrt{r^2+s^2} \}$$

Well, another phenomenal find: (p 17)

	$f(x)$	$g(y) = \int_0^\infty f(x) \cos(xy)\, dx \quad y > 0$
(27)	$(x^2 + a^2)^{-\frac{1}{2}} \exp[-\beta(x^2+a^2)^{\frac{1}{2}}]$ $\mathrm{Re}\ a > 0, \quad \mathrm{Re}\ \beta > 0$	$K_0[a(\beta^2 + y^2)^{\frac{1}{2}}]$

from which we conclude that

$$\int_0^\infty ds \cos(sy) \{ \exp[-\sqrt{r^2+s^2}\,(z+a)] / \sqrt{r^2+s^2} \} \;=\; K_0(r\sqrt{(z+a)^2 + y^2})$$

with our intermediate result

$$V(x,y,z) = \; 2^2(-q/2\pi) \int_0^\infty dr \cos(rx)\, K_0(r\sqrt{(z+a)^2 + y^2})$$

The relevant integral is now (p 49)

(40)	$K_\nu(ax)$	$\mathrm{Re}\ a > 0, \quad \lvert\mathrm{Re}\ \nu\rvert < 1$	$\frac{1}{4}(y^2+a^2)^{-\frac{1}{2}} \sec(\frac{1}{2}\nu\pi)$ $\times\{\pi a^{-\nu}[y+(y^2+a^2)^{\frac{1}{2}}]^\nu$ $+\pi a^{\nu}[y+(y^2+a^2)^{\frac{1}{2}}]^{-\nu}\}$

where $\nu = 0$ and $\alpha = \sqrt{(z+a)^2 + y^2}$. Objects with ν exponents on the right are all just 1, so $\{\} = 2\pi$ and $\sec(1/2\pi\nu) = \sec(0) = 1/\cos(0) = 1$ and all that is left is 1/4 and the first factor, so we get

$$\int_0^\infty dr \cos(rx)\, K_0(r\sqrt{(z+a)^2 + y^2}) \;=\; 2^{-2}\, 2\pi\, (\, x^2 + y^2 + (z+a)^2)^{-1/2}$$

so our answer is then

$$V(x,y,z) = 2^2(-q/2\pi)\, 2^{-2}\, 2\pi /\sqrt{x^2 + y^2 + (z+a)^2} = -q / \sqrt{x^2 + y^2 + (z+a)^2}$$

Let's pause to scan for any Bateman errata in Vol 4. Page numbers above used were 9,56,17,49 which I reorder as 9,17,49,56. But none of these numbers appears in Kok.

I would say that V(x,y,z) was the potential of a charge -q located at z = -a, which is of course the image charge size and location! If I add V(x,y,z) to my original point charge, I should get the Green's functions, and Eureka, it works. I have never done anything like this exercise before. The Bateman vol 4 was crucial, though I suppose these integrals are also somewhere in GR. Well, the GR7 has 9 entire new chapters that my old GR4 did not have, and one of these chapters is a list of transforms. The desired integrals just don't fit into the main body of a GR type book. But the cosine Fourier transforms are all there now. GR has advanced !

Problem 4: Point charge and Sphere by Dirichlet Method

I guess one could do an analogous demonstration using a sphere instead of a plane. In this case, if you put the Green's charge at some weird location, one of the two oscillatory orthogonal functions would be Legendre $P_n^m(z)$ functions, and we would find ourselves in need of a table of transforms against this function. These do not appear in GR. Even Bateman does not have such a table, the Russian Vol 5 has mainly Bessel stuff.

The starting step in this problem would be to draw a picture with +q at z = a on the + z axis. The +q/R R would be $R^2 = a^2 + r^2 - 2a\cos\theta$ so our prescribed potential on the sphere of radius b would be

$$V(b,\theta,\varphi) = -q/\sqrt{a^2 + b^2 - 2ab\cos\theta}\ .$$

Since our choice of Green's location is azisym, our Smythian atomic form would be

$$V(r,\theta,\varphi) = \Sigma_n A_n (r/b)^n P_n(\cos\theta)$$

and our Dirichlet match condition is on the sphere surface at r = b is

$$\Sigma_n A_n P_n(\cos\theta) = -q/\sqrt{a^2 + b^2 - 2ab\cos\theta} = (-q/b) / \sqrt{1 + \alpha^2 - 2\alpha\cos\theta} \qquad \alpha = a/b < 1$$

We would then use the fact that $1/\sqrt{1+\alpha^2-2\alpha\cos(\theta)} = \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n$ to get

$$\Sigma_n A_n P_n(\cos\theta) = (-q/b) \Sigma_{n=0}^{\infty} P_n(\cos\theta)\, \alpha^n$$

and I would conclude that $A_n = (-q/b)(a/b)^n$. The Dirichlet potential is then

$$V(r,\theta,\varphi) = (-q/b) \Sigma_n (r/b)^n (a/b)^n P_n(\cos\theta) = (-q/b) \Sigma_n (ra/b^2)^n P_n(\cos\theta)$$

But now $\Sigma_n(ra/b^2)^n P_n(\cos\theta) = \Sigma_n \beta^n P_n(\cos\theta) = 1/\sqrt{1+\beta^2-2\beta\cos(\theta)}$ with $\beta = ra/b^2$ so

$$V(r,\theta,\varphi) = (-q/b)/\sqrt{1+\beta^2-2\beta\cos(\theta)} = -q\,(b/a)/\sqrt{d^2+r^2-2rd\cos(\theta)} = -q\,(b/a)/R'$$

where $d \equiv b^2/a$. Draw a picture and you see that this is the potential of a point charge of size $-q\,(b/a)$ which is located a distance $d = b^2/a$ up the z axis. The distance between a point in the sphere and the d point is R'. The last step then is to add this to the original point charge and we then have the Green's Function for a sphere, and we have now derived the famous image method result using our Dirichlet method of finding a Green's function!