Mass, Charge, and Heat Flow Math PhL 10.20.09

I started pondering this subject while driving to Torrey on Oct 2, just to pass the time. I was wondering about the idea of "charge-momentum" and how continuity looked in the mass world versus the charge world. I forgot about this subject until yesterday in the shower. At the same time, this subject came up in Stakgold where he is doing physical problems which involve the Laplace equation. Basically these three sections all lead to Potential Theory of Fluid Flow, Electrostatics, and Heat Flow. Perhaps someday I will add another section relating to Diffusion. An odd fact is that in the case of charge flow, the potential theory has nothing to do with the flow, whereas it does in the other two cases.

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Overview.

The purpose here is to show the analogies between flow of mass (fluid), charge and heat, which are three somewhat different subjects, but which share a lot of math. At the end of each section, after presenting the "edited" theory, I give a list of definitions and equations and show the implications of the divergence theorem when it is a applied to a certain "flow" vector in each case. I will present those summary sections right here:

Mass Flow in a Fluid of Massive Particles:

$\mathbf{p_i} = m_i \mathbf{v_i}$	the mass-momentum of a particle
$\mathbf{P}(\mathbf{x}) = (1/V)\Sigma_{i \in v}\, m_i \mathbf{v_i}$	the macroscopic mass-momentum density of a fluid of particles
$\mathbf{k}(\mathbf{x},t) = \Sigma_{i,all}\, \delta(\mathbf{x}-\mathbf{x_i})\, m_i \mathbf{v_i}$	the microscopic mass-momentum density of a fluid
$\mathbf{P} = (1/V) \int dV\, \mathbf{k}(\mathbf{x},t)$	relation between the two mass-momentum densities
$\mathbf{P}(\mathbf{x}) = n(\mathbf{x})\, \mathbf{p_{av}}(\mathbf{x})$	definition of the average mass-momentum of a particle
$\mathbf{P}(\mathbf{x}) = \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})$	definition of the velocity vector $\mathbf{v}(\mathbf{x})$, ρ = mass density

The symbol **P** can be interpreted in two ways:

P = mass-current density	mass/sec/area
P = mass-momentum density	momentum/volume

Divergence theorem applied to **P** :

$\nabla \bullet \mathbf{P}(x) = -\partial_t \rho(\mathbf{x})$	where $\rho(x)$ is the mass density of the fluid	$\partial_\mu \mathbf{P}^\mu = 0$
$\rho(\mathbf{x}) = \Sigma_i\, m_i\, n_i(\mathbf{x})$	a sum over species, not individual particles	

If our fluid is **incompressible**, like many liquids, then $\rho(\mathbf{x})$ = constant in both space and time, and we get

$\nabla \bullet \mathbf{v}(\mathbf{x}) = 0$

If, in addition, we happen to know that ∇ x $\mathbf{v}(\mathbf{x}) = 0$ ("**irrotational** flow"), then (and only then) we can represent the velocity vector as the gradient of a **velocity potential** $\varphi(x)$ [we already have v = volume, velocity]

$\mathbf{v}(\mathbf{x}) = \pm \nabla\varphi(x)$ // some people put a minus sign here

$\Rightarrow \nabla^2\varphi(x) = 0$

and we then can apply Potential Theory to this special case of fluid flow.

Charge Flow in a Fluid of Charged Particles:

$\mathbf{s}_i = q_i\mathbf{v}_i$	the charge-momentum of a particle
$\mathbf{J}(\mathbf{x}) = (1/V)\Sigma_{i \in V}\, q_i\mathbf{v}_i$	the macroscopic charge-momentum density of a fluid of particles
$\mathbf{j}(\mathbf{x},t) = \Sigma_{i,all}\, \delta(\mathbf{x}-\mathbf{x}_i)\, q_i\mathbf{v}_i$	the microscopic charge-momentum density of a fluid
$\mathbf{J} = (1/V) \int dV\, \mathbf{j}(\mathbf{x},t)$	relation between the two charge-momentum densities
$\mathbf{J}(\mathbf{x}) = n(\mathbf{x})\, \mathbf{s}_{av}(\mathbf{x})$	definition of the average charge-momentum of a particle
$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})$	definition of the velocity vector $\mathbf{v}(\mathbf{x})$, ρ = charge-density

The symbol $\mathbf{J}$ can be interpreted in two ways:

$\mathbf{J}$ = charge-current density	charge/sec/area	"current density" in normal use
$\mathbf{J}$ = charge-momentum density	charge-momentum/volume	

Divergence theorem applied to $\mathbf{J}$:

$\nabla \bullet \mathbf{J}(x) = -\partial_t\rho(\mathbf{x})$	where $\rho(x)$ is the charge density of the fluid	$\partial_\mu \mathbf{J}^\mu = 0$
$\rho(x) = \Sigma_i\, q_i\, n_i(x)$	a sum over species, not individual particles	

In the world of Charges, Potential Theory is not really related to the flow of charge as it is related to the flow of mass in the world of Masses. Rather, we have $\nabla^2 V = \rho$ for the electrostatic potential V which is caused by charge density ρ.

Heat Flow in a Fluid of Thermal Particles:

$u_i = (1/2)\, m_i\mathbf{v}_i \bullet \mathbf{v}_i$	the kinetic energy of a particle
$U(\mathbf{x}) = (1/V)\Sigma_{i \in V}\, u_i$	the macroscopic kinetic energy density of a fluid of particles
$k(\mathbf{x},t) = \Sigma_{i,all}\, \delta(\mathbf{x}-\mathbf{x}_i)\, u_i(\mathbf{x}_i)$	the microscopic kinetic energy density of a fluid

$U = (1/V) \int dV\, k(\mathbf{x},t)$ relation between the two kinetic energy densities

$U(\mathbf{x}) = n(\mathbf{x})\, u_{av}(\mathbf{x})$ definition of the average kinetic energy of a particle

$\mathbf{F} = -k \nabla T(\mathbf{x})$ definition of heat current density

$\nabla \bullet \mathbf{F}(\mathbf{x}) = -\partial_t U(\mathbf{x})$ Gauss's Law

The symbols U and $\mathbf{F}$ can be interpreted as follows: ("heat" = "thermal energy" = "kinetic energy")

U = kinetic energy density kinetic energy/volume

$\mathbf{F}$ = heat current density heat/sec/area

Divergence theorem applied to $\mathbf{F}$:

$\nabla \bullet \mathbf{F}(x) = -\partial_t U(\mathbf{x})$ where $U(x)$ is the energy density of the fluid $\partial_\mu F^\mu = 0$

$U(\mathbf{x}) = \Sigma_i\, u_{av,i}(\mathbf{x})\, n_i(\mathbf{x})$ a sum over species, not individual particles

If we postulate this relationship between kinetic energy density, heat capacity and temperature,

$$U(\mathbf{x}) = C\, T(\mathbf{x})$$

where C is the (constant in space) heat capacity of the fluid, then applying our divergence result above, then using our $\mathbf{F}$ definition above, and assuming k(x) = k, we find

$$\nabla^2 T(\mathbf{x}) = -(1/k) f(\mathbf{x})$$

where f(x) is out heat source distribution (analogous to the charge density in the Charge case.)

A. Mass Flow

(1) Imagine we have a fluid of particles labeled by index i, each having momentum

$$\mathbf{p}_i = m_i \mathbf{v}_i$$

Consider the following quantity, where V is a volume large enough to hold many particles of our fluid, but is small relative to the dimensions of whatever "problem" we intend to work on.

$$\mathbf{P}(\mathbf{x}) = (1/V) \Sigma_{i \in V}\, m_i \mathbf{v}_i$$

Clearly this quantity is the "**macroscopic momentum density**" with dimensions (momentum/m^3). If V contains some reasonably large statistical number of particles, and if we have some external conditions (like EM fields, gravity, whatever) that vary slowly (in space) relative to perhaps the mean free path of our particles in their collision action, then we feel pretty strongly that we can write $\mathbf{P}$ as a function of $\mathbf{x}$. We could go off and spend a lot of time creating a detailed model, including temperature, which would

justify the above claim, but I am happy with the claim without such details right now. The quantity $\mathbf{P}(\mathbf{x})$ is mathematically a vector field: the **linear momentum density field**.

If all particles in our fluid had the same mass m, we could say

$$\mathbf{P}(\mathbf{x}) = m(1/V)\Sigma_{i\in V}\,\mathbf{v}_i = m\,\mathbf{V}(\mathbf{x}) \qquad\qquad \mathbf{V}(\mathbf{x}) = (1/V)\Sigma_{i\in V}\,\mathbf{v}_i$$

and of course we would call $\mathbf{V}(\mathbf{x})$ the **velocity density field** of our gas (dim = velox/m^3). I don't think anyone uses such an object, but it is at least something we can define.

(2) Now consider this quantity, where we sum over all particles in some large region of space R in which we are working. (For example, any volume V we think of will be inside R.)

$$\mathbf{k}(\mathbf{x},t) = \Sigma_{i,all}\,\delta(\mathbf{x}-\mathbf{x}_i)\,m_i\mathbf{v}_i \qquad\qquad \text{dim} = \text{mom}/m^3$$

I would call this the "**microscopic momentum density**". This vector quantity is zero except where particles exist, namely at $\mathbf{x} = \mathbf{x}_i$, and it is a distribution. Now what happens if we take the average of this thing over some small but statistical volume?

$$(1/V)\int dV\,\mathbf{k}(\mathbf{x},t) = (1/V)\,\Sigma_{i,all}\int dV\,\delta(\mathbf{x}-\mathbf{x}_i)\,m_i\mathbf{v}_i = (1/V)\,\Sigma_i\,m_i\mathbf{v}_i = \mathbf{P}$$

So we are not surprised to see that the macroscopic momentum density is the average of the microscopic momentum density.

(3) Now we want to talk about the **average momentum of a particle** in our fluid. We write

$$\mathbf{p}_{av}(\mathbf{x}) = (1/N_V)\,\Sigma_{i\in V}\,m_i\mathbf{v}_i$$

where again we are happy that this is a function of space due to external influences. Here N_V is the number of particles in our volume V. We now assume some number density $n(\mathbf{x}) = N_V/V$ so we can write $[1/N_V = 1/[Vn(\mathbf{x})]$

$$\mathbf{p}_{av}(\mathbf{x}) = (1/N_V)\,\Sigma_{i\in V}\,m_i\mathbf{v}_i = 1/[Vn(\mathbf{x})]\;\Sigma_{i\in V}\,m_i\mathbf{v}_i = (1/n(\mathbf{x})\,(1/V)\,\Sigma_{i\in V}\,m_i\mathbf{v}_i = \mathbf{P}(\mathbf{x})\,/n(\mathbf{x})$$

Therefore, again not much surprise, we find that

$$\mathbf{P}(\mathbf{x}) = n(\mathbf{x})\,\mathbf{p}_{av}(\mathbf{x})$$

The momentum density is the average particle momentum times the particle number density.

(4) Suppose we consider a tiny volume of fluid. The total momentum in our volume is $\mathbf{P}_V = \mathbf{P}(\mathbf{x})V$ and the total mass in our volume is $M_V = \rho(\mathbf{x})V$. If we think of this tiny volume as if it were a particle or at least a small object, then we can say it has a velocity defined by $\mathbf{P}_V = M_V\,\mathbf{v}$. So let's define

$\mathbf{v}(\mathbf{x}) \equiv \mathbf{P}_V/M_V = \mathbf{P}(\mathbf{x})V/ [\rho(\mathbf{x})V] = \mathbf{P}(\mathbf{x})/ \rho(\mathbf{x})$ mass/sec/m^2 * 1/[mass/m^3] = m/sec

Clearly $\mathbf{v}(\mathbf{x})$ is a "field" and has dimensions of velocity. We shall call it "the velocity of the fluid" or "the velocity field" or "**the velocity vector**". Do not confuse this with $\mathbf{V}(\mathbf{x})$ defined earlier which was the "velocity density field" for a fluid of identical mass particles. For such a fluid we would have in fact

$m\ \mathbf{V}(\mathbf{x}) = \mathbf{P}(\mathbf{x}) = \rho(\mathbf{x})\ \mathbf{v}(\mathbf{x})$ => $\mathbf{V}(\mathbf{x}) = [\rho(\mathbf{x})/m]\ \mathbf{v}(\mathbf{x}) = n(\mathbf{x})\ \mathbf{v}(\mathbf{x})$

where $n(\mathbf{x})$ is the number density of particles. The quantity $\mathbf{v}(\mathbf{x})$ has a meaning even when the masses of the particles are not the same, whereas $\mathbf{V}(\mathbf{x})$ does not.

(5) We could also *talk about* the average velocity of a particle in the fluid

$\mathbf{v}_{av}(x) = (1/V) \Sigma_i\ \mathbf{v}_i = (1/V) \Sigma_i\ (\mathbf{p}_i/m_i)$

I think this equals $\mathbf{v}(\mathbf{x})$ only if all the particles have the same mass. Suppose the fluid contains two species called a and b. Then we have

$\mathbf{v}_{a,av}(\mathbf{x}) = \mathbf{P}_a(\mathbf{x})\ /\rho_a(\mathbf{x})$ and similarly for b

$\mathbf{v}(\mathbf{x}) = \mathbf{P}(\mathbf{x})/ \rho(\mathbf{x}) = (\mathbf{P}_a(\mathbf{x}) + \mathbf{P}_b(\mathbf{x})) / (\rho_a(\mathbf{x}) + \rho_b(\mathbf{x}))$

$= \mathbf{P}_a(\mathbf{x}) / (\rho_a(\mathbf{x}) + \rho_b(\mathbf{x})) + \mathbf{P}_b(\mathbf{x}) / (\rho_a(\mathbf{x}) + \rho_b(\mathbf{x}))$

$= \mathbf{v}_{a,av}(\mathbf{x})\ \rho_a(\mathbf{x})/ (\rho_a(\mathbf{x}) + \rho_b(\mathbf{x})) + \mathbf{v}_{b,av}(\mathbf{x})\ \rho_b(\mathbf{x})/ (\rho_a(\mathbf{x}) + \rho_b(\mathbf{x}))$

$= [\ (1/V) \Sigma_{i,a}\ \mathbf{v}_i\ ]\ \rho_a(\mathbf{x})/\rho(\mathbf{x}) + [\ (1/V) \Sigma_{i,b}\ \mathbf{v}_i\ ]\ \rho_b(\mathbf{x})/ \rho(\mathbf{x})$

but I see no way to relate this to $\mathbf{v}_{av}(x) = (1/V) \Sigma_i\ \mathbf{v}_i = [\ (1/V) \Sigma_{i,a}\ \mathbf{v}_i\ ] + [\ (1/V) \Sigma_{i,b}\ \mathbf{v}_i\ ]$.

(6) Now let's review the ideas above, and I will be adding the term "mass" to various terms, anticipating the next section.

$\mathbf{p}_i = m_i\mathbf{v}_i$	the mass-momentum of a particle
$\mathbf{P}(\mathbf{x}) = (1/V)\Sigma_{i\in V}\ m_i\mathbf{v}_i$	the macroscopic mass-momentum density of a fluid of particles
$\mathbf{k}(\mathbf{x},t) = \Sigma_{i,all}\ \delta(\mathbf{x}-\mathbf{x}_i)\ m_i\mathbf{v}_i$	the microscopic mass-momentum density of a fluid
$\mathbf{P} = (1/V) \int dV\ \mathbf{k}(\mathbf{x},t)$	relation between the two mass-momentum densities
$\mathbf{P}(\mathbf{x}) = n(\mathbf{x})\ \mathbf{p}_{av}(\mathbf{x})$	definition of the average mass-momentum of a particle

$\mathbf{P}(\mathbf{x}) = \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})$ definition of the velocity vector $\mathbf{v}(\mathbf{x})$, ρ = mass density

The symbol $\mathbf{P}$ can be interpreted in two ways:

$\mathbf{P}$ = mass-current density mass/sec/area
$\mathbf{P}$ = mass-momentum density momentum/volume

Imagine a tube through which a fluid flows (called a pipe). You could talk about a mass-current through such a tube. It would be the total mass per second flowing through the tube at any cross sectional cut. If no mass is created or destroyed in the tube, the mass current would be the same at any such cross sectional cut. The word "density" appearing in the above interpretations has two different meanings: per area, and per volume. [I don't know if there is a standard symbol for mass current like I].

(7) We now apply the divergence theorem to $\mathbf{P}$ and make use of its "mass-current density" interpretation,

$$\int dV\, \nabla \bullet \mathbf{P} = \int d\mathbf{S} \bullet \mathbf{P}$$

on a closed boundary. The RHS is the total flow of mass out through the boundary and must equal the loss of total mass enclosed, assuming no mass creation or destruction. Therefore, we may conclude that

$\nabla \bullet \mathbf{P}(x) = -\partial_t \rho(\mathbf{x})$ where $\rho(x)$ is the mass density of the fluid $\partial_\mu \mathbf{P}^\mu = 0$

$\rho(\mathbf{x}) = \Sigma_i\, m_i\, n_i(\mathbf{x})$ a sum over species, not individual particles

Another way to write this makes use of the velocity vector noted above, so we then have

$$\nabla \bullet [\, \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})\,] = -\partial_t \rho(\mathbf{x})$$

and this appears as equation (3.25) on page 36 of my Schaum fluid dynamics book. If our fluid is **incompressible**, like many liquids, then $\rho(\mathbf{x})$ = constant in both space and time, and we get

$$\nabla \bullet \mathbf{v}(\mathbf{x}) = 0$$

If, in addition, we happen to know that $\nabla \times \mathbf{v}(\mathbf{x}) = 0$ ("**irrotational** flow"), then (and only then) we can represent the velocity vector as the gradient of a **velocity potential** $\varphi(x)$ [we already have v = volume, velocity]

$\mathbf{v}(\mathbf{x}) = \pm \nabla \varphi(x)$ // some people put a minus sign here

In this special case, then, we have

$\nabla^2 \varphi(x) = 0$ // Laplace

and then we can try to solve such fluid flow problems using our Stakgold potential theory methods. There are various definitions of an "ideal fluid", and here is the one I think Stakgold considers:

"In this context we define an *ideal fluid* as an incompressible fluid without viscosity."

Maybe no viscosity causes the flow to be irrotational.

B. Charge Flow

Now I want to translate Sections (7) and (8) above into charge stuff in place of mass stuff:

(6a) Review of the facts:

$\mathbf{s}_i = q_i \mathbf{v}_i$	the charge-momentum of a particle
$\mathbf{J}(\mathbf{x}) = (1/V)\Sigma_{i \in V}\, q_i \mathbf{v}_i$	the macroscopic charge-momentum density of a fluid of particles
$\mathbf{j}(\mathbf{x},t) = \Sigma_{i,all}\, \delta(\mathbf{x}-\mathbf{x}_i)\, q_i \mathbf{v}_i$	the microscopic charge-momentum density of a fluid
$\mathbf{J} = (1/V) \int dV\, \mathbf{j}(\mathbf{x},t)$	relation between the two charge-momentum densities
$\mathbf{J}(\mathbf{x}) = n(\mathbf{x})\, \mathbf{s}_{av}(\mathbf{x})$	definition of the average charge-momentum of a particle
$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})$	definition of the velocity vector $\mathbf{v}(\mathbf{x})$, ρ = charge-density

The symbol $\mathbf{J}$ can be interpreted in two ways:

$\mathbf{J}$ = charge-current density	charge/sec/area	"current density" in normal use
$\mathbf{J}$ = charge-momentum density	charge-momentum/volume	

Imagine a tube through which a fluid flows (called a wire). You could talk about a charge-current through such a tube. It would be the total charge per second flowing through the tube at any cross sectional cut. If no charge is created or destroyed in the tube, the charge current would be the same at any such cross sectional cut. The word "density" appearing in the above interpretations has two different meanings: per area, and per volume. [The charge-current is usually called I.]

(7a) We now apply the divergence theorem to $\mathbf{J}$ and make use of its "charge-current density" interpretation,

$$\int dV\, \nabla \bullet \mathbf{J} = \int d\mathbf{S} \bullet \mathbf{J}$$

on a closed boundary. The RHS is the total flow of charge out through the boundary and must equal the loss of total charge enclosed, assuming no charge creation or destruction. Therefore, we may conclude that

$\nabla \bullet \mathbf{J}(x) = -\partial_t \rho(\mathbf{x})$ where $\rho(x)$ is the charge density of the fluid $\partial_\mu J^\mu = 0$

$\rho(x) = \Sigma_i q_i n_i(x)$ a sum over species, not individual particles

Another way to write this makes use of the velocity vector noted above, so we then have

$\nabla \bullet [\, \rho(\mathbf{x})\, \mathbf{v}(\mathbf{x})\,] = -\partial_t \rho(\mathbf{x})$

and this appears nowhere I know about! If our fluid is incompressible -- well here we have to stop. I don't know of any situation where charge density is "incompressible". I don't know how you would enforce the condition $\rho(\mathbf{x})$ = constant in some region of interest. So the rest of section 7 just does not apply. We will never define a velocity potential in this context.

So we don't get a Laplace equation for the velocity potential in this context. But, there is of course the *electrostatic* potential $V(\mathbf{x})$ and it satisfies the Laplace equation in charge-free space $\nabla^2 V = 0$ and the gradient of this potential is a vector field called **E**, not the velocity vector **v**.

Section added 8.12.10: Can we apply the above work to "heat flow" ?

C. Heat Flow

This is I think an interesting question. In the above cases of mass flow and charge flow, each particle had a specific and fixed mass or charge m_i or q_i which does not change. What could change is the number density of particles $n(\mathbf{x})$. When we talk "heat", a particle has some microscopic kinetic energy $u_i = (1/2)m_i\langle v_i^2\rangle$ where $\langle\rangle$ is the average in thermal equilibrium, that is, it is an average over the Maxwellian speed distribution. This is the "heat" that a particle carries, and it does not seem particularly "fixed" the way q_i or m_i are fixed. But let's try to mimic the original discussion above for mass current. I will delete sections that don't make any sense in this new context.

(1) Imagine we have a fluid of particles labeled by index i, each having kinetic energy

$u_i = (1/2)\, m_i\langle v_i^2\rangle$ // was $\mathbf{p}_i = m_i \mathbf{v}_i$

Consider the following quantity, where V is a volume large enough to hold many particles of our fluid, but is small relative to the dimensions of whatever "problem" we intend to work on.

$U(\mathbf{x}) = (1/V)\Sigma_{i \in V}\ (1/2)\, m_i\langle v_i^2\rangle = (1/V)\Sigma_{i \in V}\, u_i$

Clearly this quantity is the "**macroscopic kinetic energy density**" with dimensions (energy/m^3). If V contains some reasonably large statistical number of particles, and if we have some external conditions (like EM fields, gravity, whatever) that vary slowly (in space) relative to perhaps the mean free path of

our particles in their collision action, then we feel pretty strongly that we can write U as a function of $\mathbf{x}$. We could go off and spend a lot of time creating a detailed model, including temperature, which would justify the above claim, but I am happy with the claim without such details right now. The quantity $U(\mathbf{x})$ is mathematically a vector field: the **kinetic energy density field**.

(2) Now consider this quantity, where we sum over all particles in some large region of space R in which we are working. (For example, any volume V we think of will be inside R.)

$$k(\mathbf{x},t) = \Sigma_{i,all}\, \delta(\mathbf{x}-\mathbf{x}_i)\, m_i v_i(\mathbf{x}_i)^2 \qquad \dim = \text{mom}/m^3$$

I would call this the "**microscopic kinetic energy density**". This scalar quantity is zero except where particles exist, namely at $\mathbf{x} = \mathbf{x}_i$, and it is a distribution. Now what happens if we take the average of this thing over some small but statistical volume,

$$(1/V) \int dV\, k(\mathbf{x},t) = (1/V)\, \Sigma_{i,all} \int dV\, \delta(\mathbf{x}-\mathbf{x}_i)\, u_i = (1/V)\, \Sigma_i\, m_i v_i(\mathbf{x}_i)^2$$

If we then take a thermal equilibrium average of the above we get

$$(1/V) < \int dV\, k(\mathbf{x},t) > = (1/V) < \Sigma_{i,all} \int dV\, \delta(\mathbf{x}-\mathbf{x}_i)\, u_i > = (1/V)\, \Sigma_i\, m_i\, < v_i(\mathbf{x}_i)^2 > = U(\mathbf{x})$$

So we are not surprised to see that the macroscopic kinetic energy density is the double-average of the microscopic kinetic energy density.

(3) Now we want to talk about the **average kinetic energy of a particle** in our fluid. We write

$$u_{av}(\mathbf{x}) = (1/N_V)\, \Sigma_{i \in V}\, u_i \qquad \text{// Note that i just labels particles, is not a species label}$$

where again we are happy that this is a function of space due to external influences. Here N_V is the number of particles in our volume V. We now assume some number density $n(\mathbf{x}) = N_V/V$ so we can write $[1/N_V = 1/[Vn(\mathbf{x})]$

$$u_{av}(\mathbf{x}) = (1/N_V)\, \Sigma_{i \in V}\, u_i = 1/[Vn(\mathbf{x})]\ \Sigma_{i \in V}\, u_i = (1/n(\mathbf{x})\, (1/V)\, \Sigma_{i \in V}\, u_i = U(\mathbf{x})\, /n(\mathbf{x})$$

Therefore, again not much surprise, we find that

$$U(\mathbf{x}) = n(\mathbf{x})\, u_{av}(\mathbf{x})$$

The kinetic energy density is the average particle kinetic energy times the particle number density.

However, I suspect that in fact we have

$$u_{av}(\mathbf{x}) = u_i = (1/2)\, m_i < v_i^2 >$$

because probably every point in our tiny volume has the same Maxell distribution. This result by the way is for one species only. I think we could generalize to multiple species this way

$$u_{av}(\mathbf{x}) = (1/n_{tot}(x))\Sigma_{ij}\, n_j(\mathbf{x})\, u_{i,j} = (1/n_{tot}(x))\Sigma_{ij}\, n_j(\mathbf{x})\,(1/2)\, m_{i,j}\langle v_i^2\rangle_j$$

where $n_{tot}(x) = \Sigma_j\, n_j(\mathbf{x})$.

Comment: We are still thinking of our medium here as a fluid of particles bounding thermally around. More generally we can have other "modes" which each store kT/2 of energy and have nothing to do with the mass of free particles, but have analogous forms such as (1/2) I ω^2 for a rotation say. I am just trying here to copy down items from the mass and charge sections above.

(4) This item does not seem to make much sense, so skip it.

Comment before item (5). In the mass and charge cases, the particles actually moved, bringing their mass or charge "with them". The movement of interest there was really the macroscopic mass or charge *momentum* density, not the random movement of individual particles. But on average, actual charge or mass really did "move" in position. In our pipes, mass or charge definitely flows down the pipe.

In the heat case, things are not so clear. We imagine that "heat" can move without movement of the macroscopic volumes. This is because the thermal energy is transferred from cell to cell, so it moves but the "average particle" need not move. On the other hand, we could also apply a "from cell to cell" view of the motion of mass or charge. In that model, a particle moves from cell 1 to cell 2, then some other particle moves cell 2 to cell 3, it is "ten little Indians all in a bed". In this model, the charge or mass is "transferred" from cell to cell, and in the aggregate, we see overall "motion" as if particles moved freely. However, in our heat case, we don't even have particles moving a single cell? What then do we mean by saying "the heat or KE is transferred from cell to cell"? This can be done without mass transfer.

How do we model this? In our most basic model, we talk about a piece of area dA and a piece of time dt and we say (dQ/dt)/dA = k ΔT where dQ/dt/dA is somehow our "heat flow". Can we relate this somehow to $U(\mathbf{x})$? This model (imagine it is staged left to right) implies that something is moving to the right, being *driven* by the temperature difference. We need some kind of flow vector here. But we don't a momentum flow vector, because we know we can transfer heat without transferring momentum. So I don't want to copy down from above anything that involves momentum, so maybe totally new item (5):

(5) The total KE in some volume V is $\int dV\, U(\mathbf{x})$. We can define an "energy flow vector $\mathbf{F}$ " such that the following would be true (F is the energy flow "out" of the volume, for a sphere $\mathbf{F} = F\hat{\mathbf{r}}$ **say)**

$$d/dt\,[\int dV\, U(\mathbf{x})\,] = -\int d\mathbf{A}\bullet \mathbf{F} \qquad \Rightarrow \qquad \nabla\bullet \mathbf{F}(\mathbf{x}) = -\partial_t U(\mathbf{x})$$

so that if the sum of all the energy inflows and outflows relative to our volume is 0, then the total KE inside that volume remains constant. Presumably energy flow is driven by temperature differences, so I would say then that

$\mathbf{F} = -k\,\nabla T(\mathbf{x})$ which says that our KE flows "downhill" (ignoring entropy S and all that stuff)

$\Rightarrow d\mathbf{A}\bullet \mathbf{F} = -k\, d\mathbf{A}\bullet\nabla T(\mathbf{x}) = -k\, dA\, \partial_{\mathbf{n}} T(x)$

where $\partial_{\mathbf{n}}$ means the change in temperature normal to a point on the surface. This is now starting to look like Stak page 327 D where we would refer to $d\left[\int dV\, U(\mathbf{x})\right] = -dQ$ where we put a - sign because out flowing $\mathbf{F}$ creates a negative $-\int d\mathbf{A}\bullet \mathbf{F}$ and thus a negative $d/dt\left[\int dV\, U(\mathbf{x})\right]$, but we want dQ to be the heat the flowing "out" (to match Stak) which is positive. Then we have

$dQ = -d\left[\int dV\, U(\mathbf{x})\right] = dt \int d\mathbf{A}\bullet \mathbf{F} = dt \int [-k\, dA\, \partial_{\mathbf{n}} T(x)]$ // matches p 327 D

so we can identify k with the "thermal conductivity" of the Stak discussion.

How do we interpret $\mathbf{F}$ and what are its units? From the Gauss integral statement above, it seems pretty clear that $d\mathbf{A}\bullet\mathbf{F}$ = energy /sec. If we write $\mathbf{F} = F\,\hat{\mathbf{n}}$, then $dA_{\mathbf{n}}\, F$ = energy/sec and F = energy/sec/area. This is the thing we always write as dQ/dt/dA.

So $\mathbf{F}$ is what we have been seeing, a thermal energy flow vector.

(6) Now let's review the ideas above, and I will be adding the term "kinetic energy" to various terms, anticipating the next section.

$u_{\mathbf{i}} = (1/2)\, m_{\mathbf{i}} \mathbf{v}_{\mathbf{i}} \bullet \mathbf{v}_{\mathbf{i}}$	the kinetic energy of a particle
$U(\mathbf{x}) = (1/V)\Sigma_{\mathbf{i}\in\mathbf{v}}\, u_{\mathbf{i}}$	the macroscopic kinetic energy density of a fluid of particles
$k(\mathbf{x},t) = \Sigma_{\mathbf{i},\mathbf{all}}\, \delta(\mathbf{x}-\mathbf{x}_{\mathbf{i}})\, u_{\mathbf{i}}(\mathbf{x}_{\mathbf{i}})$	the microscopic kinetic energy density of a fluid
$U = (1/V) \int dV\, k(\mathbf{x},t)$	relation between the two kinetic energy densities
$U(\mathbf{x}) = n(\mathbf{x})\, u_{\mathbf{av}}(\mathbf{x})$	definition of the average kinetic energy of a particle
$\mathbf{F} = -k\, \nabla T(\mathbf{x})$	definition of heat current density
$\nabla\bullet \mathbf{F}(\mathbf{x}) = -\partial_{\mathbf{t}} U(\mathbf{x})$	Gauss's Law

The symbols U and $\mathbf{F}$ can be interpreted as follows: ("heat" = "thermal energy" = "kinetic energy")

U = kinetic energy density	kinetic energy/volume
$\mathbf{F}$ = heat current density	heat/sec/area

Imagine a tube through which heat flows (called a pipe). You could talk about a heat-current through such a tube. It would be the total heat per second flowing through the tube at any cross sectional cut. If no heat is created or destroyed in the tube (insulated walls and no sources), the heat current would be the same at any such cross sectional cut. [I don't know if there is a standard symbol for heat current like I].

(7) We now apply the divergence theorem to **F** and make use of its "heat-current density" interpretation,

$$\int dV\ \nabla \bullet \mathbf{F} = \int d\mathbf{S} \bullet \mathbf{F}$$

on a closed boundary. The RHS is the total flow of energy out through the boundary and must equal the loss of total energy enclosed, assuming no mass creation or destruction. Therefore, we may conclude that

$\nabla \bullet \mathbf{F}\ (x) = -\partial_t U(\mathbf{x})$ where $U\ (x)$ is the energy density of the fluid $\qquad \partial_\mu F^\mu = 0$

$U(\mathbf{x}) = \Sigma_i\ u_{av,i}(\mathbf{x})\ n_i(\mathbf{x})$ a sum over species, not individual particles

where now for the first time I allow several "species" of particles and allow that they might have different average thermal energy.

Another way to write this makes use of the velocity vector noted above, so we then have

$$\nabla \bullet [k(\mathbf{x})\ \nabla T(\mathbf{x})\ ] = \partial_t U(\mathbf{x})$$

In our work above, we assumed there were no heat "sources" *inside* our volume. Obviously if there are, they will add to $\partial_t U(\mathbf{x})$ and we will then have

$$\partial_t U(\mathbf{x}) = f(\mathbf{x}) + \nabla \bullet [k(\mathbf{x})\ \nabla T(\mathbf{x})\ ]$$

where f(x) represents a sum of our sources and has units energy/volume/sec.

Now for the first time we use some extra information, namely

$$U(\mathbf{x}) = C(\mathbf{x})T(\mathbf{x})$$

which says the thermal energy density is proportional to temperature and C is the total heat capacity of our mixed species fluid. C has units energy/volume/degree. For a unit volume, if you raise the temperature by 1 degree (Kelvin) then U increases by C. A large heat capacity causes a large increase in U. So this "new fact" comes in completely from outside our discussion here. Since we have a fluid, we cannot really have $C = C(\mathbf{x})$, so C is a constant.

But, now we bring in another "new" idea. Our system of particles might not be a fluid. In our charge discussion above, our medium might be some non-uniform metal alloy, and might thus be a solid. So if we allow as our medium here is possibly a solid, then we can have $C = C(\mathbf{x})$ so that this property might vary with position in the non-uniform solid. We don't expect to have $C = C(\mathbf{x},t)$ however.

Similarly we don't expect $k = k(\mathbf{x},t)$. Putting in the time dependences we now have

$$C(\mathbf{x})\ \partial_t T(\mathbf{x},t) = f(\mathbf{x},t) + \nabla \bullet [k(\mathbf{x})\ \nabla T(\mathbf{x},t)\ ]$$

We can write the above as

$\partial_t T(\mathbf{x}) - (1/C(\mathbf{x}))\nabla \bullet [k(\mathbf{x})\,\nabla T(\mathbf{x})] = (1/C(\mathbf{x}))\, f(\mathbf{x},t)$ // this is Stak p 328 A.10

Although I have not gotten that far yet in Stak, this is obviously the PDE (equation of evolution) for heat flow.

This "heat flow" is not radiation or convection, it is conduction, so the above we have the PDE for heat conduction as named in Stak.

Suppose k(x) is uniform in the medium, and suppose we are in a static situation. Then we have

$$-(k/C(\mathbf{x}))\nabla \bullet \nabla T(\mathbf{x}) = (1/C(\mathbf{x}))\, f(\mathbf{x})$$

or

$$-(k/C(\mathbf{x}))\nabla^2 T(\mathbf{x}) = (1/C(\mathbf{x}))\, f(\mathbf{x})$$

or

$$-k\nabla^2 T(\mathbf{x}) = f(\mathbf{x})$$

or

$$\nabla^2 T(\mathbf{x}) = -(1/k) f(\mathbf{x})$$

and finally we have arrived at the fact that, in this special case, T(x) solves the Poisson equation and we can regard T(**x**) as a "potential" and the RHS as the "source" which drives this potential. The analogy to the "electric field" would be $-\nabla T(x) = (1/k)\mathbf{F}$ so we can think of $(1/k)\mathbf{F}$ as **E** if we like.