Stakgold Exercise 6.47 PhL 10.23.09

Here we are going to actually solve a real integral equation 6.190 with a log type kernel. Without peeking, my usual approach would be to diagonalize it with an appropriate transform. You need a set of basis functions that you know how to integrate against the log. You could differentiate to get rid of the log somehow. But I think Stak is going to give us an *ad hoc* method. He does a change of variables, and THEN he does a basis function expansion. There are lots of tricky jogs in his path that I would not have taken were I on my own. (1) I would have left sinβ in the integral, which messes things up. (2) taking sinβ into F(β), I would have done a sine transform on F(β). Both these wrong turns lead to death.

(a) Derive 6.191

We start with:

$$\int_{-1}^{1} dy\, f(y) \ln|x-y| = g(x)$$

First he wants to change variables this way

$x = \cos\alpha \qquad y = \cos\beta$

so we get $y = -1$ means $\beta = \pi$ and $y = +1$ means $\beta = 0$ and also $dy = -\sin\beta$, so

$$\int_{0}^{\pi} d\beta \sin\beta\, f(\cos\beta) \ln|\cos\alpha - \cos\beta| = g(\cos\alpha)$$

Now replace $F(\beta) = f(\cos\beta)\sin\beta$ and $G(\alpha) = g(\cos\alpha)$,

$$\int_{0}^{\pi} d\beta\, F(\beta) \ln|\cos\alpha - \cos\beta| = G(\alpha)$$ **// which is 6.191**

(b) Derive 6.192

Let's start from the other end and see how 6.22 must look,

$$\Sigma_{n=1} (1/n) \cos(n\alpha)\cos(n\beta) = (1/2) \Sigma_{n=1} (1/n) \{ \cos(n\alpha-n\beta) + \cos(n\alpha+n\beta) \}$$

$$= (1/2) \Sigma_{n=1} (1/n) \cos(n\alpha-n\beta) + (1/2) \Sigma_{n=1} (1/n) \cos(n\alpha+n\beta)$$

Now set $\alpha' = 1$ in 6.22 and we get, using 6.22 twice,

$$= (1/2)(-1/2) \ln[2-2\cos(\alpha-\beta)] + (1/2)(-1/2) \ln[2-2\cos(\alpha+\beta)]$$

$$= (-1/4) \ln[2-2\cos(\alpha-\beta)] + (-1/4) \ln[2-2\cos(\alpha+\beta)]$$

Now set $[2-2\cos(\alpha-\beta)] = 4\ \{\ [1-\cos(\alpha-\beta)]/2\ \} = 4\sin^2[(\alpha-\beta)/2]$ so we have

$$\ln[2-2\cos(\alpha-\beta)] = \ln\ \{\ 4\sin^2[(\alpha-\beta)/2]\ \} = 2\ln 2 + \ln\sin^2[(\alpha-\beta)/2]$$

So we have now shown that

$$\Sigma_{n=1}\ (1/n)\cos(n\alpha)\cos(n\beta) = (-1/4)\ \{\ 2\ln 2 + \ln\sin^2[(\alpha+\beta)/2]\} + (-1/4)\ \{\ 2\ln 2 + \ln\sin^2[(\alpha-\beta)/2]\}$$

$$= -\ln 2 + (-1/4)\ln\ \{\ \sin^2[(\alpha+\beta)/2]\sin^2[(\alpha-\beta)/2]\ \}$$

$$= -\ln 2 + (-1/2)\ln|\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]|$$

$$= -\ln 2 + (-1/2)\ln[\ |\cos\alpha - \cos\beta|)/2\ ] \qquad \text{Schaum p 17 5.64}$$

$$= -\ln 2 + (-1/2)\ln|\cos\alpha - \cos\beta|) + (+1/2)\ln 2$$

$$= -(\ln 2)/2 + (-1/2)\ln|\cos\alpha - \cos\beta|)$$

Therefore we have

$$\ln|\cos\alpha - \cos\beta| = -2\ \Sigma_{n=1}\ (1/n)\cos(n\alpha)\cos(n\beta) - \ln 2 \qquad \textbf{// which is 6.192}$$

So this is going to be used to convert our kernel in 6.191 into a trig form. Notice the "separable" form of the kernel, although we have an infinite sum. Now go back to the above form

$$\Sigma_{n=1}\ (1/n)\cos(n\alpha)\cos(n\beta) = -(\ln 2)/2 + (-1/2)\ln|\cos\alpha - \cos\beta|)$$

$$(-1/\pi)\ \Sigma_{n=1}\ (1/n)\cos(n\alpha)\cos(n\beta) = (+1/2\pi)\ [\ln 2 + \ln|\cos\alpha - \cos\beta|\ ] \equiv m(\alpha,\beta)$$

and so we have also verified **6.188**.

(c) Derive 6.193.

We are told to expand F and G on "cosine series". We have from above that

$$F(\beta) = f(\cos\beta)\sin\beta \qquad 0,\pi$$

$$G(\beta) = g(\cos\beta) \qquad 0,\pi$$

Our angle range of interest is $(0,\pi)$, so we are allowed to do "as we like" outside this range. Suppose then we consider the range $(-\pi,\pi)$ but we force F to be even

$$F(\beta) = f(\cos\beta)\,|\sin\beta| = f(\cos\beta)\sqrt{1-\cos^2\beta} \qquad (-\pi,\pi)$$

$G(\beta) = g(\cos\beta)$ $(-\pi,\pi)$

Now we can do a "cosine series" expansion on both these functions, to wit [Schaum p 131 2L = 2π]

$F(\beta) = \Sigma_{m=1}\, f_m \cos(m\beta) + f_0/2$ $\quad f_m = (1/\pi) \int_{-\pi}^{\pi} F(\beta) \cos(m\beta)\, d\beta = (2/\pi) \int_{0}^{\pi} F(\beta) \cos(m\beta)\, d\beta$

$G(\beta) = \Sigma_{m=1}\, g_m \cos(m\beta) + g_0/2$ $\quad g_m = (1/\pi) \int_{-\pi}^{\pi} G(\beta) \cos(m\beta)\, d\beta = (2/\pi) \int_{0}^{\pi} G(\beta) \cos(m\beta)\, d\beta$

Now here is our equation of interest, where on the second line we use our previous identity:

$$\int_0^{\pi} d\beta \ \ln|\cos\alpha - \cos\beta| F(\beta) = G(\alpha)$$

$$\int_0^{\pi} d\beta \ \{-2 \Sigma_{n=1} (1/n) \cos(n\alpha)\cos(n\beta) - \ln 2\} F(\beta) = G(\alpha)$$

We are hoping this thing will "diagonalize". Let's then apply $(2/\pi) \int_0^{\pi} d\alpha \cos(m\alpha)$ to both sides:

$$g_m = (2/\pi) \int_0^{\pi} d\alpha \cos(m\alpha) \int_0^{\pi} d\beta \ \{-2 \Sigma_{n=1} (1/n) \cos(n\alpha)\cos(n\beta) - \ln 2\} F(\beta)$$

Now in the ln2 term we have all by itself

$$\int_0^{\pi} d\alpha \cos(m\alpha) = (1/m)\sin(m\alpha)|^{\pi}_0 = 0 \qquad m \neq 0$$

$$= \pi \qquad m = 0$$

$$= \pi\, \delta_{m,0}$$

so this term in g_m has this value

$$g_m \text{ (the ln2 term)} = (2/\pi)\, \pi\, \delta_{m,0}\, (-\ln 2) \int_0^{\pi} d\beta\, F(\beta) = (2/\pi)\, \pi\, \delta_{m,0}\, (-\ln 2)\, (\pi f_0/2)$$

$$= \pi\, \delta_{m,0}\, (-\ln 2)\, (f_0) = -\pi \ln 2\, f_0\, \delta_{m,0}$$

So this ln2 term will only affect the g_0 equation. Now the *other* term in g_m is

$$(2/\pi) \int_0^{\pi} d\alpha \cos(m\alpha) \int_0^{\pi} d\beta \ \{-2 \Sigma_{n=1} (1/n) \cos(n\alpha)\cos(n\beta)\} F(\beta)$$

$$= (-4/\pi) \Sigma_{n=1} (1/n) \int_0^{\pi} d\beta \cos(n\beta) F(\beta) \int_0^{\pi} d\alpha \cos(m\alpha) \cos(n\alpha)$$

$= (-4/\pi) \Sigma_{n=1} (1/n) \int_0^{\pi} d\beta \cos(n\beta) F(\beta)(\pi\delta_{m,n}/2)$ $= 0$ if m = 0, else:

$= (-2) (1/m) \int_0^{\pi} d\beta \cos(n\beta) F(\beta) = (-2/m) \pi f_m/2 = (- \pi f_m/m)$

We have now found that

$g_0 = -\pi \ln 2 \, f_0$
$g_m = (- \pi f_m/m)$ m = 1,2...

which we can solve to get

$f_0 = - (1/\pi \ln 2) \, g_0$
$f_m = -(m/\pi) g_m$

Now we can then write:

$F(\beta) = \Sigma_{m=1} f_m \cos(m\beta) + f_0/2$

$= - \Sigma_{m=1} \cos(m\beta) (m/\pi) g_m - (1/2) (1/\pi \ln 2) \, g_0$

$= - \Sigma_{m=1} \cos(m\beta) (m/\pi) (2/\pi) \int_0^{\pi} G(\alpha) \cos(m\alpha) \, d\alpha - (1/2) (1/\pi \ln 2) (2/\pi) \int_0^{\pi} G(\alpha) \, d\alpha$

$= - (1/\pi^2 \ln 2) \int_0^{\pi} G(\alpha) \, d\alpha - (2/\pi^2) \Sigma_{m=1} m \cos(m\beta) \int_0^{\pi} G(\alpha) \cos(m\alpha) \, d\alpha$

which at last is **6.193.**

<u>(c) Derive 6.194.</u>

Start with B, derive C:

$F(\beta) = (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha \, G(\alpha) - (2/\pi^2) \Sigma_{n=1} n \cos(n\beta) \int_0^{\pi} d\alpha \, G(\alpha) \cos(n\alpha)$

Let's try the trick he used earlier to say

$-n \cos(n\beta) = \partial_\beta^2 \cos(n\beta)/n$

so the above becomes

$F(\beta) = (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha \, G(\alpha) + (2/\pi^2) \partial_\beta^2 \Sigma_{n=1} (1/n) \cos(n\beta) \int_0^{\pi} d\alpha \, G(\alpha) \cos(n\alpha)$

$$= (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha\, G(\alpha) + (2/\pi^2)\, \partial_\beta^2 \int_0^{\pi} d\alpha\, G(\alpha)\, \{\Sigma_{n=1} (1/n) \cos(n\beta) \cos(n\alpha)\}$$

Now we can use our identity

$$\Sigma_{n=1} (1/n) \cos(n\alpha)\cos(n\beta) = - \ln 2 + (-1/2) \ln | \cos\alpha - \cos\beta |$$

to get

$$F(\beta) == (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha\, G(\alpha) + (2/\pi^2)\, \partial_\beta^2 \int_0^{\pi} d\alpha\, G(\alpha)\, \{- \ln 2 + (-1/2) \ln | \cos\alpha - \cos\beta | \}$$

Now the ln2 term gives nothing under ∂_β^2 so have now

$$F(\beta) = (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha\, G(\alpha) + (2/\pi^2)\, (-1/2)\, \partial_\beta^2 \int_0^{\pi} d\alpha\, G(\alpha) \ln | \cos\alpha - \cos\beta |$$

$$= (-1/\pi^2 \ln 2) \int_0^{\pi} d\alpha\, G(\alpha) - (1/\pi^2)\, \partial_\beta^2 \int_0^{\pi} d\alpha\, G(\alpha) \ln | \cos\alpha - \cos\beta |$$

and this is his **result C.**

(d) Derive 6.195.

Suppose g(x) = 1. Then G(β) = 1 and we have (we just did this integral above)

$$g_m = (2/\pi) \int_0^{\pi} 1 \cos(m\beta)\, d\beta = (2/\pi)\, \pi\, \delta_{m,0} = 2\, \delta_{m,0}$$

Then we have

$$f_0 = - (1/\pi \ln 2)\, g_0 = - (2/\pi \ln 2)$$
$$f_m = -(m/\pi) g_m = 0$$

and then

$$F(\beta) = \Sigma_{m=1} f_m \cos(m\beta) + f_0/2 = f_0/2 = - (1/\pi \ln 2)$$

But from above we had

$$F(\beta) = f(\cos\beta) | \sin\beta | = f(\cos\beta) \sqrt{1-\cos^2\beta} \qquad (-\pi,\pi)$$

Therefore

$f(\cos\beta)\sqrt{1-\cos^2\beta} = -(1/\pi\ln 2)$

$f(x)\sqrt{1-x^2} = -(1/\pi\ln 2)$

$f(x) = -(1/\pi\ln 2)/\sqrt{1-x^2}$ **which is 6.195**

Comments on this problem:

We know that we can diagonalize any equation of this form:

$$a(g_1) = \int dg\, b(g_1 g^{-1})\, c(g)$$

where dg is the invariant Haar measure of the group of interest. I think we need $\int dg = 1$ but I forget why. Here is the "proof" of the diagonalization:

$a^\sigma \equiv \int dg\, D^\sigma(g) a(g)$ // the items on the left and D are all square matrices

$b^\sigma \equiv \int dg\, D^\sigma(g) b(g)$

$c^\sigma \equiv \int dg\, D^\sigma(g) c(g)$

$a(g_1) = \int dg\, b(g_1 g^{-1})\, c(g)$

$a^\sigma = \int dg_1\, D_\sigma(g_1) a(g_1) = \int dg_1\, D_\sigma(g_1) \{ \int dg\, b(g_1 g^{-1})\, c(g) \}$

$= \int dg \{ \int dg_1\, D_\sigma(g_1)\, b(g_1 g^{-1}) \}\, c(g)$

$= \int dg \{ \int dg_1\, D_\sigma(g_1 g^{-1})\, D_\sigma(g)\, b(g_1 g^{-1}) \}\, c(g)$ // D is a group representation

$= \int dg \{ \int d(g_1 g^{-1}) (D_\sigma(g_1 g^{-1})\, D_\sigma(g)\, b(g_1 g^{-1}) \}\, c(g)$ // rearrangement theorem

$= \int dg \{ \int dg_2 (D_\sigma(g_2)\, D_\sigma(g)\, b(g_2) \}\, c(g)$

$= \int dg_2\, D_\sigma(g_2)\, b(g_2) \int dg\, D_\sigma(g)\, c(g)$

$= b^\sigma\, c^\sigma$

so we end up with this matrix diagonalization where σ is a rep label

$a^\sigma = b^\sigma\, c^\sigma$

The D functions have to be properly normalized. For the abelian SO(2) group we have (σ = integer) only one dimensional representations,

$$D_\sigma(g) = e^{i\theta\sigma} \qquad D_\sigma(g)D_\sigma(g^{-1}) = e^{i\theta\sigma}\, e^{i(-\theta\sigma)} = 1$$

$$a^\sigma = \int dg\, D^\sigma(g)a(g) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, e^{i\sigma\theta}\, a(\theta)$$

If a,b,c are all real and even functions of θ, then we have

$$a^\sigma = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, e^{i\sigma\theta}\, a(\theta) = (1/2\pi) \int_{-\pi}^{\pi} d\theta \cos(\sigma\theta)\, a(\theta) = (1/\pi) \int_{0}^{\pi} d\theta \cos(\sigma\theta)\, a(\theta)$$

So our convolution equation is this and its diagonalization is on the next line

$$a(\alpha) = (1/2\pi) \int d\beta\, g(\alpha-\beta)c(\beta)$$

$$a_\sigma = g_\sigma\, c_\sigma \qquad \text{// diagonalization}$$

In our application we had a = G, c = F and according to the following, b = $\{\pi \ln | \cos\alpha - \cos\beta |\}$:

$$G(\alpha) = (1/2) \int_{-\pi}^{\pi} d\beta \ln | \cos\alpha - \cos\beta |\, F(\beta)$$

$$= (1/2\pi) \int_{-\pi}^{\pi} d\beta\, \{\pi \ln | \cos\alpha - \cos\beta |\}\, F(\beta)$$

Our result was this:

$$g_0 = (-\pi \ln 2)\, f_0$$
$$g_m = (-\pi/m)\, f_m \qquad m = 1,2...$$

I think we can conclude that

$$b_0 = (-\pi \ln 2)$$
$$b_m = (-\pi/m)$$

Somewhere from 30+ years ago I have good notes on this, Stakgold did not get involved in group interpretations of things.