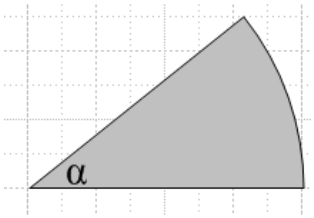
Stakgold Exercise 6.48 PhL 10.28.09

I was not successful in finding the answer here after ~12 hours of work. I started off in Section A solving the entirely wrong problem! Then in Section B I started on the right method using the integral equations. I could see that this was going to work, so I paused it about half way through. I thought I was then going to find a faster solution in Section C doing a conformal map method, but that failed after lots of hours. So if I decide I need to know the solution to this problem, I will come back here and finish Section B. I would do that now, but I know from experience that I will make algebraic errors which will have to be fixed in multiple passes and it will likely take another 10 hours to do, not worth it.

Exercise 6.48.

Put some charge on a metal 2D wedge and figure out how it distributes itself, compute the potential everywhere outside the wedge, etc.



Off hand, I sort of expect charge to peak up on the sharp corners.

A. Treated as the Wrong Problem, aborted after a while (but learned a few things so doing)

I think I already know the Green's function for this pup? Nope, we never did it. We did a Dirichlet on it with f(φ) on the rim in Ex 6.10, then we did the eigenfunctions in 6.28 (which weren't pretty). Maybe the integral equation method will be easier than doing the Green's.

What about mapping this thing into the unit circle and just using the known solution there? Suppose we make this change of variable

$$s = f(z) = z^{2\pi/\alpha}$$

which would make the above thing be a <u>disk in s-space</u>. Set radius to 1 for now. I don't thing there is a "slot problem" here if we go all the way and just seal the gap on the positive real z axis. Hmmm.

Does such a variable change preserve harmonicity? Odd that this simple issue is not mentioned directly in Ahlfors or Stakgold. The claim is clearly stated in Theorems 2 and 3 p 233 of my Schaum complex variable book, and he proves it in problems 3 and 4. So this is a Big Point that I just missed, though I knew it was true. Not only is harmonicity preserved, but a constant boundary value is also preserved.

Now what is Green's for the unit disk? It's the charge plus image charge thing, 6.100 for example, or this more apt result from the conformal section,

$$g(\mathbf{r}|\mathbf{r}_0) = -(1/2\pi)\ln|z-z_0| + (1/2\pi)\ln|z-z_0^*| + (1/2\pi)\ln|z_0|$$

Now we really want this in s-space for use above, so we have

$$g(\mathbf{s}|\mathbf{s}_0) = -(1/2\pi)\ln|s-s_0| + (1/2\pi)\ln|s-s_0^*| + (1/2\pi)\ln|s_0|$$

where $\mathbf{s} = (p, q)$ and $s = p + iq$. Now I guess all we have to do is install the transform to get

$$g(\mathbf{z}|\boldsymbol{\xi}) = -(1/2\pi)\ln|z^{2\pi/\alpha} - \xi^{2\pi/\alpha}| + (1/2\pi)\ln|z^{2\pi/\alpha} - (\xi^*)^{2\pi/\alpha}| + (1/2\pi)\ln|\xi^{2\pi/\alpha}|$$

where $\xi^* = 1^2/\xi$ since radius is 1

where the positive main unit point charge is at ξ and we observe at z. I cannot call this x because the gets us confused with the x of x + iy. Let's write this in polar coordinates where $z = r\,e^{i\theta}$ and $\xi = \rho e^{i\varphi}$:

$$g(r,\theta \mid \rho,\varphi) = -(1/2\pi)\ln|z^{2\pi/\alpha} - \xi^{2\pi/\alpha}| + (1/2\pi)\ln|z^{2\pi/\alpha} - \xi^{-2\pi/\alpha}| + (1/2\pi)\ln|\xi^{2\pi/\alpha}|$$

where I just imagine putting these forms in for z and ξ. Our charge density is $\Sigma = -\partial_{\mathbf{r}}\, g(r,\theta \mid \rho,\varphi)$ at least on the outer edge. We will need a different derivative on the straight edges, hold on that for a while. I am going to assume that this expression is both the internal and the external Green's Function. To get the charge density on the arc, I want to approach from the outside. We need to expose r-dependence in order to do the derivative. So

$$g(r,\theta \mid \rho,\varphi) = -(1/2\pi)\ln|r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha}| + (1/2\pi)\ln|r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha}| + (1/2\pi)\ln|\xi^{2\pi/\alpha}|$$

$$= -(1/4\pi)\ln(r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha}) + c.c$$

$$+ (1/4\pi)\ln(r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha}) + c.c.$$

$$+ (1/2\pi)\ln|\xi^{2\pi/\alpha}|$$

$$\partial_{\mathbf{r}}\, g(r,\theta \mid \rho,\varphi) = -(1/4\pi)(r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha})^{-1}(2\pi/\alpha)\,r^{2\pi/\alpha-1}\,e^{i2\pi\theta/\alpha} + c.c$$

$$+ (1/4\pi)(r^{2\pi/\alpha}e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha})^{-1}(2\pi/\alpha)\,r^{2\pi/\alpha-1}\,e^{i2\pi\theta/\alpha} + c.c$$

Now set r = 1 since that is where the charge is on the arc'd part

$$\partial_{\mathbf{r}}\, g(r,\theta \mid \rho,\varphi)\big|^{\mathbf{r=1}} = -(1/4\pi)(e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha})^{-1}(2\pi/\alpha)\,e^{i2\pi\theta/\alpha} + c.c$$

$$+ (1/4\pi)(e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha})^{-1}(2\pi/\alpha)\,e^{i2\pi\theta/\alpha} + c.c$$

$$= -(1/2\alpha)\{e^{i2\pi\theta/\alpha}/(e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha})\} + c.c$$
$$+(1/2\alpha)\{e^{i2\pi\theta/\alpha}/(e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha})\} + c.c$$

$$= 2\,\mathrm{Re}\{-(1/2\alpha)\{e^{i2\pi\theta/\alpha}/(e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha})\}$$
$$+ 2\,\mathrm{Re}\{+(1/2\alpha)\{e^{i2\pi\theta/\alpha}/(e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha})\}$$

$$= (1/\alpha)\,[\; - \mathrm{Re}\,\{\, e^{i2\pi\theta/\alpha} \,/\, (e^{i2\pi\theta/\alpha} - \xi^{2\pi/\alpha}) \,\} + \mathrm{Re}\;\{\, e^{i2\pi\theta/\alpha} \,/\, (e^{i2\pi\theta/\alpha} - \xi^{-2\pi/\alpha}) \,\}$$

where $\xi = \rho e^{i\varphi}$. I am expecting this thing to blow up at $\varphi = \alpha$ but does not seem so.

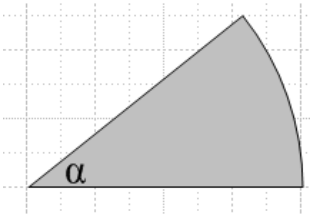
OUCH! I am doing the wrong problem! Above I am computing the charge on the wedge arc that is induced due to the presence of a Green's point charge nearby at location ξ. An interesting problem perhaps but now what was asked. We are asked to deposit charge Q on the wedge and tell how it distributes itself!

B. The integral equation method.

Our first kind Fred for this problem is

$$0 = c_0 + \int_C d\ell_\xi \;\; E(s|\xi)\,\Sigma(\xi) \qquad\qquad E\,\Sigma = -c_0 \qquad \text{first kind Fred}$$

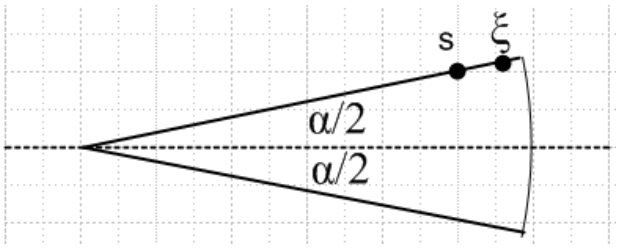
We can think of our wedge perimeter as a "bent circle". I can do this problem two ways:



(1) find a conformal map that takes the wedge to the circle, then transform the circle result to get the potential for the wedge, then compute derivatives to get the charges. I am suspicious of the mapping I used above, worried about the slot, it is not a smooth morphing! Better maybe to take this wedge to the half plane, then take it to the unit circle.

(2) write out then solve the integral equation!

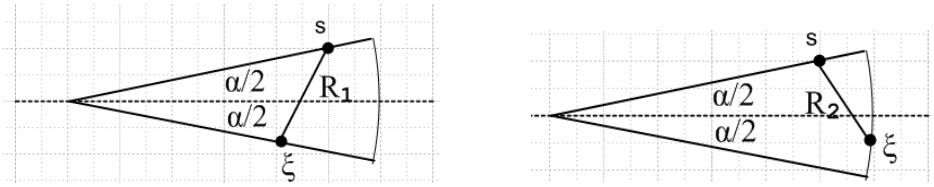
Let's save the conformal map method for C below, and try at least to set up the integral equation stuff. Here is a picture showing the symmetry of the situation



For each position of point s, there are three different edges on which ξ can be located. The σ integral for this position of s shown here would be { $E = (-1/2\pi)\ln R$ } { Let $\beta = \alpha/2$ to save space }

$$(-2\pi)\int_C d\ell_\xi \; E(s|\xi)\,\Sigma(\xi) = \int_0^1 dr \; \ln|r-r_s|\,\Sigma(r,\beta) + \int_0^1 dr \ln R_1 \; \Sigma(r,-\beta)$$

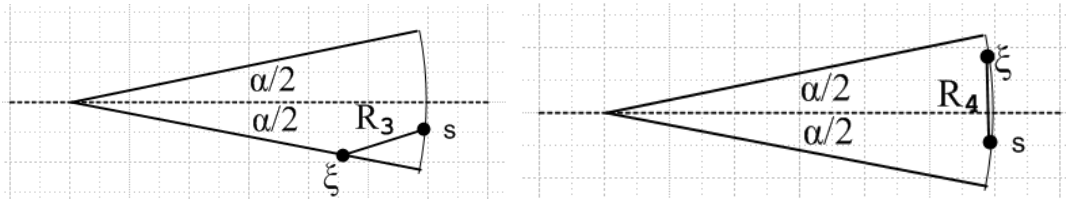
$$+ \int_{-\beta}^{\beta} 1 d\varphi \; \ln R_2 \; \Sigma(1,\varphi)$$



If we put s on the lower straight edge, nothing new can really be learned, given the symmetry. But we could put s on the arc and then we would get

$$(-2\pi)\int_C d\ell_\xi \; E(s|\xi)\,\Sigma(\xi) = \int_0^1 dr \ln R_2 \; \Sigma(r,\beta) + \int_0^1 dr \ln R_3 \; \Sigma(r,-\beta)$$

$$+ \int_{-\beta}^{\beta} 1 d\varphi \ln R_4 \; \Sigma(1,\varphi)$$



There are really two distinct surface charge functions which I could define as follows:

$f(r) = \Sigma(r,\beta) = \Sigma(r,-\beta)$ // due to symmetry

$g(\varphi) = \Sigma(1,\varphi) = \Sigma(1,-\varphi)$ // symmetric in φ

Let's renormalize our equations as follows

$$0 = (-2\pi)c_0 + (-2\pi)\int_C d\ell_\xi \; E(s|\xi)\,\Sigma(\xi)$$

$$\Rightarrow 0 = C_0 + (-2\pi)\int_C d\ell_\xi \; E(s|\xi)\,\Sigma(\xi) \qquad C_0 = -2\pi c_0$$

Then our two integral equations are:

$$-C_0 = \int_0^1 dr \; \ln|r-r_s|\,\Sigma(r,\alpha/2) + \int_0^1 dr \ln R_1 \Sigma(r,-\alpha/2) + \int_{-\beta}^{\beta} 1 d\varphi \; \ln R_2 \Sigma(1,\varphi)$$

$$-C_0 = \int_0^1 dr\, \ln R_2 \Sigma(r,\alpha/2) + \int_0^1 dr\, \ln R_3 \Sigma(r,-\alpha/2) + \int_{-\beta}^{\beta} 1 d\varphi\, \ln R_4\ \Sigma(1,\varphi)$$

$$-C_0 = \int_0^1 dr\ \ln|r-r_s|\, f(r) + \int_0^1 dr\, \ln R_1\, f(r) + \int_{-\beta}^{\beta} 1 d\varphi\, \ln R_2\, g(\varphi)$$

$$-C_0 = \int_0^1 dr\, \ln R_2\, f(r) + \int_0^1 dr\, \ln R_3\, f(r) + \int_{-\beta}^{\beta} 1 d\varphi\, \ln R_4\ g(\varphi)$$

Now about the various R_i, let's try to use laws of cosines:

$$R_1^2 = r_s^2 + r_\xi^2 - 2 r_s r_\xi \cos(\alpha) = r_s^2 + r^2 - 2\, r_s\ r \cos(\alpha)$$

$$R_2^2 = r_s^2 + 1^2 - 2\, r_s \cos(\beta - \varphi)$$

$$R_3^2 = r^2 + 1^2 - 2\, r \cos(\beta + \varphi)$$

$$R_4^2 = 1^2 + 1^2 - 2 \cos(\varphi - \varphi_s)$$

Now what do we do with such a horrible mess? In each of our two C_0 equations above, the RHS will be a function of r_s and φ_s, but the LHS is a constant, so somehow this must work out.

Compute the R_1 integral. Let's look at one particular integral to gain some insight,

$$\int_0^1 dr\, \ln R_1\, f(r) = \int_0^1 dr\, f(r)\, \{ (1/2)\ \ln[\, r_s^2 + r^2 - 2\, r_s\ r \cos(\alpha)] \ \}$$

We are going to have to use 6.22 many times here to get "separable form" for things. In this case,

$$(1/2) \ln[\, r_s^2 + r^2 - 2\, r_s\ r \cos(\alpha)]$$

is our quantity of interest. In 6.22 we now overload the symbol α and say

$$1 + \alpha^2 = r_s^2 + r^2 \qquad \alpha = r_s\ r$$

but this tells us the formula cannot be used, ouch! Let's try to message things a bit

$$\ln[\, r_s^2 + r^2 - 2\, r_s\ r \cos(\alpha)] = \ln\{ [\, r_s^2] [\, 1 + (r/r_s)^2 - 2\, (r/r_s) \cos(\alpha)]\}$$

$$= 2 \ln r_s + \ln [\, 1 + (r/r_s)^2 - 2\, (r/r_s) \cos(\alpha)]$$

Now we CAN use 6.22 with $\alpha = r/r_s$. So we have

$$(1/2)\ \ln[\, r_s^2 + r^2 - 2\, r_s\ r \cos(\alpha)] = \ln r_s + (1/2) \ln [\, 1 + (r/r_s)^2 - 2\, (r/r_s) \cos(\alpha)]$$

$$= \ln r_s - \Sigma_{n=1}^{\infty}\ (1/n)\, (r/r_s)^n \cos(n\alpha)$$

and so

$$\int_0^1 dr\, \ln R_1\, f(r) = \int_0^1 dr\, f(r)\, [\ln r_s - \Sigma_{n=1}^{\infty} (1/n)\, (r/r_s)^n \cos(n\alpha)\,]$$

$$= \ln r_s \int_0^1 dr\, f(r) - \Sigma_{n=1}^{\infty} (1/n) \cos(n\alpha)\, r_s^{-n} \int_0^1 dr\, f(r)\, r^n$$

Maybe as a shorthand we an introduce the "moments" of f(r) as follows

$$f_n = \int_0^1 dr\, f(r)\, r^n \qquad\qquad n = 0,1,2...$$

Then we have shown that

$$\int_0^1 dr\, \ln R_1\, f(r) = f_0 \ln r_s - \Sigma_{n=1}^{\infty} (f_n/n) \cos(n\alpha)\, r_s^{-n}$$

which is not TOO bad. But there are 5 more integrals to worry about! But we can see a little now how this is going to play out. We might get terms like $\ln(r_s)$ and r_s^m and $\cos(\varphi_s)$, who knows. But the coefficient of each term must be 0 because LHS = constant!

<u>Compute the "first integral".</u>

$$\int_0^1 dr\, \ln|r-r_s|\, f(r) = \int_0^1 dr \{ (1/2) \ln (r-r_s)^2 \}\, f(r)$$

Write

$$\ln (r-r_s)^2 = \ln [\, r_s^2 (1 - (r/r_s)^2] = 2 \ln(r_s) + \ln[1 - (r/r_s)^2]$$

Then expand using $\ln(1+x) = - \Sigma_{n=1}^{\infty} (-x)^n/n$ with $x = - (r/r_s)^2$:

$$\ln[1 - (r/r_s)^2] = - \Sigma_{n=1}^{\infty} (r/r_s)^{2n}/n$$

Then

$$(1/2) \ln (r-r_s)^2 = \ln(r_s) - \Sigma_{n=1}^{\infty} (r/r_s)^{2n}/n$$

and so

$$\int_0^1 dr\, \ln|r-r_s|\, f(r) = \int_0^1 dr\, f(r) \{ \ln(r_s) - \Sigma_{n=1}^{\infty} (r/r_s)^{2n}/n \}$$

$$= f_0 \ln(r_s) - \Sigma_{n=1}^{\infty} r_s^{-2n}(1/n) \int_0^1 dr\, f(r)\, r^{2n}$$

$$= f_0 \ln(r_s) - \Sigma_{n=1}^{\infty} r_s^{-2n}(1/n)\, f_{2n}$$

Compute the angle R_2 integral.

$$\int_{-\beta}^{\beta} 1 d\varphi \ln R_2\, g(\varphi) = 2(1/2) \int_0^{\beta} d\varphi\, g(\varphi) \ln [\, R_2^2]$$

$$= 2 \int_0^{\beta} d\varphi\, g(\varphi) \{ (1/2) \ln [\, r_s^2 + 1^2 - 2\, r_s \cos(\beta - \varphi)\,] \}$$

As before we write

$$[\, r_s^2 + 1^2 - 2\, r_s \cos(\beta - \varphi)\,] = [\, r_s^2] [\, 1 + (1/r_s)^2 - 2\, (1/r_s) \cos(\beta-\varphi)\,]$$

$$(1/2) [\, r_s^2 + 1^2 - 2\, r_s \cos(\beta - \varphi)\,] = \ln(r_s) + (1/2) \ln [\, 1 + (1/r_s)^2 - 2\, (1/r_s) \cos(\beta-\varphi)\,]$$

Then using 6.22 again we have

$$(1/2) \ln [\, 1 + (1/r_s)^2 - 2\, (1/r_s) \cos(\beta-\varphi)\,] = - \Sigma_{n=1}^{\infty} (1/n)\, (1/r_s)^n \cos(n[\beta-\varphi])$$

Therefore

$$\int_{-\beta}^{\beta} 1 d\varphi \ln R_2\, g(\varphi) = 2 \int_0^{\beta} d\varphi\, g(\varphi) \{ \ln(r_s) - \Sigma_{n=1}^{\infty} (1/n)\, (1/r_s)^n \cos(n[\beta-\varphi]) \}$$

$$= 2\ln(r_s) \int_0^{\beta} d\varphi\, g(\varphi) - \Sigma_{n=1}^{\infty} (1/n)\, (1/r_s)^n \int_0^{\beta} d\varphi\, g(\varphi) \cos(n[\beta-\varphi])$$

Now suppose we define g's moments like this:

$$g_n = \int_0^{\beta} d\varphi\, g(\varphi) \cos(n[\beta-\varphi])$$

Then this last result is

$$\int_{-\beta}^{\beta} 1 d\varphi \ln R_2\, g(\varphi) = 2\ln(r_s) g_0 - \Sigma_{n=1}^{\infty} (g_n/n)\, (1/r_s)^n$$

which fits into our general pattern.

I see how this is going to end. There are three more integrals to compute. In each of our two equations, we set the coefficient of $\ln(r_s)$ to 0 and the coefficient of r_s^{-n} to 0. This gives us two equations for the two

unknowns g_n and f_n which we can then solve. We will transform back to get our desired charge densities f and g. Constant C_0 will be set by our overall charge Q. We are doing a multi-form diagonalization here. Let's put this approach on hold for now, since I think I know how to do the problem. It was a good problem to look at for sure.

Closing comment: the conformal mapping method below was not successful for me, so if you really wanted to know the answer for this problem, you would finish what I have started in this section. But alas, I have burned a 12 hour day already on this problem, I think that is due diligence, lets move on.

C. The conformal mapping method.

For the unit metal ring, the charge is uniformly distributed and we know the potential to be this

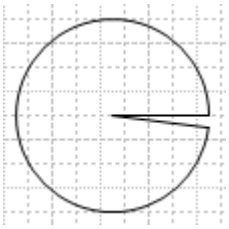
$$u = (-1/2\pi) \ln(r/1) = (-1/2\pi) \ln |z|$$

which vanishes on the ring and diverges at infinity. I then write it in complex notation.

Now for the conformal map. In section A above I considered

$$s = f(z) = z^{2\pi/\alpha}$$

which seems to map a wedge in z-space to a full unit circle in s-space. I have some doubts about this mapping due to the "slit" that occurs as $\alpha \rightarrow 2\pi$,



Things just look singular to my intuition. Suppose instead we get out our little catalog of conformal maps in the Schaum book. On page 297 we see a sector mapping from wedge to half plane in the case that the angle has the form $\alpha = \pi/m$. The wedge boundary maps into the real axis as shown. Then the half plane to circle we also know. So let's examine these:

wedge to upper half plane: $w = (1+z^m)^2/ (1-z^m)^2$ $\alpha = \pi/m$

I calculate inverse: $z = [(w^{1/2} - 1)/ (w^{1/2} + 1)]^{1/m}$

The wedge should really be called a circle sector, but OK. Then we have

upper half plane to unit disk: $s = (w-z_0)/(w-\bar{z}_0)$

I calculate inverse: $w = (s\,\bar{z}_0 - z_0)/(s-1)$

where we can pick any point in the upper half plane we like, such as $z_0 = i$, in which case we have

$s = (w-i)/(w+i)$ $\qquad$ $w = (-si - i)/(s-1) = -i\,(s+1)/(s-1)$

Now if I want to go all the way from wedge to unit disk I get

$s = (w-i)/(w+i)$ $\qquad$ where $\quad w = (1+z^m)^2/(1-z^m)^2$

disk←wedge = disk ← UHP ← wedge $\qquad$ $s \leftarrow w \leftarrow z$

Now, in the space with the unit disk, which is s-space, we have our potential being

$u = (-1/2\pi) \ln|s|$

So I expect the potential in the space which has the wedge (which is z-space) to be

$u = (-1/2\pi) \ln|(w-i)/(w+i)|$ where $\quad w = (1+z^m)^2/(1-z^m)^2$ $\quad$ and $m = \pi/\alpha$

I wonder it this is right? I certainly would like to see some contours! They don't look too good, so let's back up and examine each mapping and do some contours there.

wedge to upper half plane: $\qquad w = (1+z^m)^2/(1-z^m)^2 \qquad \alpha = \pi/m$

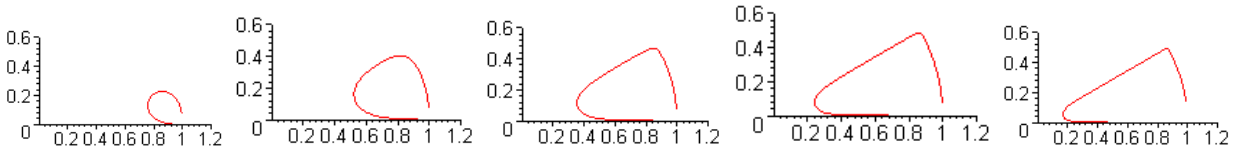
upper half plane to wedge: $\qquad z = [(w^{1/2} - 1)/(w^{1/2} + 1)]^{1/m}$

I would like to plot the latter for w = u+ib. Here is some code and an encouraging plot:

```
> m := 6:
> z := (  (sqrt(w)-1)/(sqrt(w)+1)  )^(1/m):
> w := u + I*v:
> v := .1:
> with(plots):
> complexplot(z,u=-20..20, numpoints = 1000,scaling=CONSTRAINED, view=[0..1.2, 0..0.6]);
```

If we let w be a set of horizontal lines in the UHP, what do we see in the z plane? I can just try a few values of v and see what we get:

v=1 $\qquad$ .1 $\qquad$ .01 $\qquad$ .001 $\qquad$ .0001



One could I think argue that as we lower the level curves in the w plane, we are approaching the boundary of our wedge from the inside. Very encouraging.

Now what we really want is the outside of the wedge. At first I thought that the mapping on Schaum p 207 would work such that the lower half plane maps to the outside of the wedge. But playing around makes he think the lower half plane really maps into a mirror wedge below the x axis! So if this is true, I have a major misunderstanding of these mappings.

Learning more about mappings.

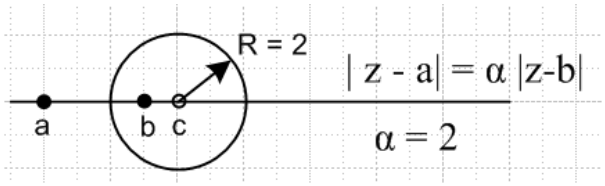
Consider mapping A-5 which takes upper half disk to upper half plane. First of all, how do I know this mapping is correct?

$w = [\ (1+z)/(1-z)]^2$ $\qquad\qquad$ $z = (\sqrt{w}\ -1)/\ (\sqrt{w}\ +1)$

The first equation tells me

$|w| = |1+z|^2\ /\ \ |1-z|^2\ \ =\ \ |z+1|^2\ /\ |z-1|^2$

Consider a circle in the w plane $|w| = K$. This implies $|z+1|\ /\ |z-1|\ = \sqrt{K}$. But Apollonius told me this is circle in the z plane with a = -1, b = 1 and $\alpha = \sqrt{K}$. Thus makes some circle in the z plane like this



I think this is true and there is no contradiction, but it does not help us with our mapping verification. Instead, it is more helpful to do this:

$z = (\sqrt{w}\ -1)/\ (\sqrt{w}\ +1)$

$w = re^{i\theta}$ $\qquad\qquad$ $|z|^2\ = [\ r+1-2\sqrt{r}\ \cos(\theta/2)]\ /\ [\ r+1+2\sqrt{r}\ \cos(\theta/2)]$ $\qquad\qquad$ // hand calc

For the upper half plane in w-space we have $\theta < \pi$ so $\theta/2 \le \pi/2$ so $\cos(\theta/2) > 0$ so clearly $|z| < 1$. This suggests that the upper half plane in w-space maps to something inside the unit circle in z-space. If we rationalize the above z = .. equation we get

$z = \ [\ r-1+2i\sqrt{r}\ \sin(\theta/2)]\ /\ [\ r+1+2\sqrt{r}\ \cos(\theta/2)]$

$Im(z) = 2\sqrt{r}\ \ \sin(\theta/2)\ /\ [\ r+1+2\sqrt{r}\ \cos(\theta/2)]\ \ > 0$

so now we know it maps into something inside the upper half circle. I will accept that it fills the entire upper half circle.

Now, where does the lower half plane map? Just think of this as $-\pi < \theta/2 < 0$. This does not affect the sign of $\cos(\theta/2)$ so we still map into the $|z| < 1$ space. Now $\sin(\theta/2) < 0$ and we get the lower half circle.

OK, after all this fiddling it seems pretty clear that you must "mirror" the two sides of $w = f(z)$ and get the obvious conclusion. It is obvious now, was not obvious earlier.

Conclusion: if you know that $R \rightarrow S$ according to some $f(z)$, you cannot conclude that $R' \rightarrow S'$ where prime means complement region.

Example: From A-5 we have mapping from upper half disk to UHP, call this $R \rightarrow S$. The mapping which takes $R' \rightarrow S$ is shown in A-4 and seems pretty unrelated to the first mapping.

THEREFORE: If I want to do this Exercise 6-48 by conformal mapping, I need a mapping that takes the *outside* of the wedge to something useful, and so A-6 does not help me at all. And such a mapping is not in my little table, but I might find it somewhere. // A quick tour did not turn up any large tables.

Let's ponder how you would connect A-4 and A-5. If we look at the A-5 upper half disk, I would think this mapping would take it to the exterior upper space

$$s = 1/z$$

So how about this sequence

s	z	w
exterior space	upper half disk	upper half plane

$$s = 1/z \qquad w = [(1+z)/(1-z)]^2$$
$$z = 1/s \qquad z = (\sqrt{w} - 1)/(\sqrt{w} + 1)$$

Then you would think that the mapping from w to s would be

$$s = (\sqrt{w} + 1)/(\sqrt{w} - 1) \qquad w = [(s+1)/(s-1)]^2$$

I think this is one viable mapping, but it is not a very nice mapping! Remember that for any pair of regions there are an infinite number of maps.

STOP. I am sure a mapping exists, but after several hours of web hunting and doing a new doc on this subject (wedge, math/ahlfors area) I have given up. I though there would be some simple easily locatable mapping, but I cannot locate it, and I don't know how to derive it for myself. I think the basic idea would be to map the circular sector exterior into the exterior of a semi-infinite rectangle exterior using $w = \ln(z)$, then find a mapping that takes that rectangle exterior to the half plane or unit circle. That would be the plan, but my time is up on this approach. 6PM 10.29.09