A Green's Function Limit PhL 10.25.09

This is a detail from Stakgold Chapter 6 meta meta notes. I have conjectured here a certain limit of a general Green's Function, and want to investigate whether or not this conjecture is true. I start with a quote from the meta-meta notes, then try an example.

• Back to our solution to Problem One and its solution:

$$u(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \qquad (*)$$

Suppose we take the limit x→ s. Since g = 0 on the boundary, the T1 term becomes 0. It must then be true that (since we know that u(s) = f(s) on the boundary σ)

$$\lim_{x\to s} \left[- \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n}\, g(x|\xi) \right] = f(s).$$

Stakgold never mentions this and I would have to investigate to understand why this is so. One could study this in the context of 6.11 on p 94, doing a limit r→ 1. The claim seems to be this

$$\lim_{x\to s} [\partial_{\xi n}\, g(\mathbf{x}|\boldsymbol{\xi})] = -\,\delta^{(n-1)}(\mathbf{s}-\boldsymbol{\xi}) \qquad s \text{ on } \alpha$$

Stakgold has certainly given us the tools to show such a fact, but he did not do it. This is not something you can check in 1D. Basically I have proven this already because (*) above has been proven and this follows from it. The dimensions of both sides are correct.

Example: Let's see what happens with the unit circle! My raw 1 notes circa p 102 say this:

$$I(r,\psi) = I(x|\xi) = -\,\partial_{\xi n}\, g(x|\xi) = -\,\partial_r\, g(r,\psi)$$

$$I(r,\psi) = 1/2\pi + 1/\pi \sum_{n=1}^{\infty} r^n \cos(n\psi) = 1/2\pi * [\ 1 - r^2\ ] / (1 + r^2 - 2r\cos\psi)$$

So what we would want to show is that

$$\lim_{r\to 1} [-I(r,\psi)] = -\,\delta^{(1)}(\mathbf{s}-\boldsymbol{\xi}) \qquad s \text{ on } \alpha$$

$$= -\,\delta(\psi) \qquad \text{I think in polar coordinates}$$

At least that is my claim for what the delta should be. If we start with 6.11 instead, it is easier to see:

$$\lim_{r\to 1} [\ (1 - r^2) / (1 + r^2 - 2r\cos(\varphi-\psi)\ ) = 2\pi\delta(\psi-\varphi)$$

$$\lim_{r\to 1} [\ (1 - r^2) / (1 + r^2 - 2r\cos\psi) = 2\pi\delta(\psi)$$

We can now set $\cos\psi = x$.

You can see that if $x \neq 1$, we get $(1 - r^2) / (1 + 1-2x) = (1 - r^2) / (2-2x) \rightarrow 0$ which is promising. At $x = 1$, the denominator is $(1-r)^2$ and then we have $(1 - r^2) / (1-r)^2 = (1+r)/(1-r) = 2/(1-r) = \infty$, and this too is promising, it looks like a delta function.

So here is what we need to show:

$$\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} d\psi \lim_{r\to 1} [(1 - r^2) / (1 + r^2-2r\cos\psi) = 2\pi$$

I think we did this in some Stak exercise, but let's do it again. Assume we can back out the r limit so that the LHS is

$$\text{LHS} = \lim_{\varepsilon\to 0} \lim_{r\to 1} (1 - r^2) \int_{-\varepsilon}^{\varepsilon} d\psi \ / (1 + r^2-2r\cos\psi)$$

Maple does the integral like so, (call the integral J)

```
Int(1/ (1 + r^2 - 2*r*cos(psi)),psi = -epsilon..+epsilon);
```

$$f := \int_{-\varepsilon}^{\varepsilon} \frac{1}{1+r^2-2r\cos(\psi)}\, d\psi$$

```
int(1/ (1 + r^2 - 2*r*cos(psi)),psi = -epsilon..+epsilon);
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$$\frac{4 \arctan\left(\frac{\tan\left(\frac{1}{2}\varepsilon\right)(1+r^2+2r)}{\sqrt{(1+r^2-2r)(1+r^2+2r)}} \right)}{\sqrt{(-1+r^2)^2}}$$

If I manually take the limit $r\rightarrow 1$ from below, this becomes

$$J = 2 \tan^{-1}\{ 2 \tan(\varepsilon/2) / (1-r) \} / (1-r)$$

$$= 2 * \pi/2 / (1-r)$$

Then we have

$$\text{LHS} = \lim_{\varepsilon\to 0} \lim_{r\to 1} 2 (1-r)\ J = \lim_{\varepsilon\to 0} \lim_{r\to 1} 2 (1-r)\ J * 2 * \pi/2 / (1-r)$$

$$= \lim_{\varepsilon\to 0} \lim_{r\to 1} 2\pi = 2\pi \qquad \text{QED}$$

Back to our conjecture:

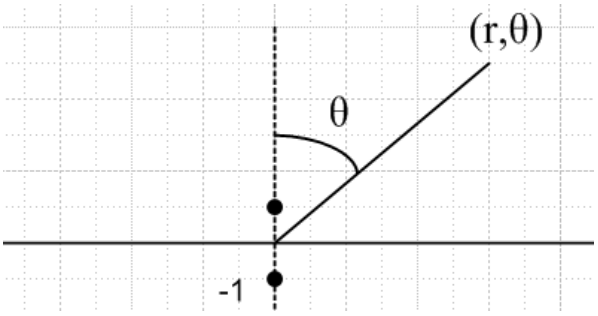
$$\lim_{x\to s} [\partial_{\xi n}\, g(x|\xi)] = - \delta^{(n-1)}(s-\xi) \qquad s \text{ on } \alpha$$

In our Problem Two discussed in meta-meta, we learned that $\partial_{\xi n}\, g(\mathbf{x}|\xi)$ is the induced charge density on the inner surface σ due to a point charge at x inside σ. So our limit concerns what happens when you let that internal point charge move to the surface! Just before contact with the surface, the surface charge at the contact point on the surface s grows very large without limit. This is because that charge is basically the normal electric field at it is getting infinite. And of course it is negative, hence our minus sign.

So maybe here is a way to think of it. As point charge at x approaches the surface we can think of the surface as basically flat, use the tangent plane. then the potential is modeled by our charge and its image charge of opposite sign. As we do the approach, we get a dipole in the limit. The dipole potential is this

$$V(r,\theta) = (1*\varepsilon)\cos\theta/r^2$$

where charge is 1 and distance to surface is ε. Observation point is at (r,θ). For a point on the surface, we have $\theta = \pi/2$ so $\cos\theta = 0$ and $V = 0$. Here is a picture



So what is the gradient of this thing (to get the electric field, hence to get the nearby charge density)

$$\nabla V(r,\theta) = \hat{\mathbf{r}}\,\partial_r V + (1/r)\partial_\theta V\,\hat{\boldsymbol{\theta}}$$

We are going to care about the $\hat{\boldsymbol{\theta}}$ component of this gradient when $\theta \to \pi/2$ so

$$\partial_\theta V = (1*\varepsilon)/r^2 * -\sin\theta$$

and therefore $E_r = (1/r)\,\partial_\theta V \to 0$ on the surface.

So this is pretty good. We see why the surface charge away from the contact point will be 0, and we see why it will be infinite at the contact point.

So here then is my interpretation of this limit

$$\lim_{x\to s}\,[\partial_{\xi n}\, g(\mathbf{x}|\xi)] = -\,\delta^{(n-1)}(\mathbf{s}-\xi) \qquad s \text{ on } \alpha$$

$$\Sigma(\text{at } \xi \text{ due to point charge at x as } x\to s \text{ on } \sigma) = -\,\delta^{\sigma}(\mathbf{s}-\xi)$$

$$\lim_{x\to s}\,\Sigma(\xi\,|x) = -\,\delta^{\sigma}(\mathbf{s}-\xi) \qquad \text{or just} \qquad \Sigma(\xi\,|s) = -\,\delta^{\sigma}(\mathbf{s}-\xi)$$

Section added on 11.18.13 while writing Disaster doc in the lines effort.

2a. Side Topic added: What happens in the $u_I(x)$ equation as $x \rightarrow s$ on boundary? It seems that then $g(s|\xi) = 0$ in the first term and it must be that

$$-(1/\varepsilon)\int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \;\rightarrow + f(x)$$

which seems to say that

$$-(1/\varepsilon)\partial_{\xi n} g(x|\xi) \rightarrow \delta^\sigma(x-\xi)$$

where $\delta^\sigma(x-\xi)$ is some kind of delta function in the dimension of the boundary. I did look this question while reading Stak (see doc in Stak Ch 6 support folder) from which I quote (with $\varepsilon = 1$),

$$\lim_{\mathbf{x}\rightarrow\mathbf{s}} [\partial_{\xi n}\, g(\mathbf{x}|\boldsymbol{\xi})] = -\,\delta^{(n-1)}(\mathbf{s}-\boldsymbol{\xi}) \qquad s \text{ on } \alpha$$

I recall that Stak never (to my knowledge) addressed this little equation. So this is a little unresolved side problem: Given

$$-\nabla^2 g(x|\xi) = \delta(x-\xi)$$

in n dimensions, show that

$$\lim_{\mathbf{x}\rightarrow\mathbf{s}} [\partial_{\xi n}\, g(\mathbf{x}|\boldsymbol{\xi})] = -\,\delta^{(n-1)}(\mathbf{s}-\boldsymbol{\xi})$$

Well, how does this work for the free space propagator in 3D where $g = 1/(4\pi R)$?

$$g(\mathbf{x}|\boldsymbol{\xi}) = 1/(4\pi|\mathbf{x}-\boldsymbol{\xi}|)$$

We have $\boldsymbol{\xi}$ on the sphere, and we want $\mathbf{x} \rightarrow \mathbf{s}$ on the sphere. We can see that $\partial_{\xi n} = \partial_r$ if we think of $\boldsymbol{\xi} = r\hat{\mathbf{r}}$ since r is then the normal direction. I guess we can say

$$\boldsymbol{\xi} = r(\sin\theta\cos\varphi\, \hat{\mathbf{x}} + \sin\theta\sin\varphi\, \hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$$
$$\mathbf{x} = r_1(\sin\theta_s\cos\varphi_s\, \hat{\mathbf{x}} + \sin\theta_s\sin\varphi_s\, \hat{\mathbf{y}} + \cos\theta_s\hat{\mathbf{z}}) \qquad r_1 < r$$

No less of generality if we pick $\mathbf{x}$ = north pole, then we have

$$\boldsymbol{\xi} = r(\sin\theta\cos\varphi\, \hat{\mathbf{x}} + \sin\theta\sin\varphi\, \hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$$
$$\mathbf{x} = r_1\hat{\mathbf{z}}$$

$$|\,\mathbf{x}-\boldsymbol{\xi}\,|^2 = x^2 + \xi^2 - 2\mathbf{x}\bullet\boldsymbol{\xi} = r_1^2 + r^2 - 2rr_1\cos\theta$$

$$| \mathbf{x}\text{-}\boldsymbol{\xi} | = \sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}$$

Note in passing that

$$\partial_{\mathbf{r}} | \mathbf{x}\text{-}\boldsymbol{\xi} | = (1/2) | \mathbf{x}\text{-}\boldsymbol{\xi} |^{-1} (2r - 2r_1\cos\theta) = | \mathbf{x}\text{-}\boldsymbol{\xi} |^{-1} (r - r_1\cos\theta)$$

Next,

$$[\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(\mathbf{x}|\boldsymbol{\xi})] = \partial_{\mathbf{r}} \{ (1/4\pi) |\mathbf{x}\text{-}\boldsymbol{\xi}|^{-1} \} = - (1/4\pi) |\mathbf{x}\text{-}\boldsymbol{\xi}|^{-2} \partial_{\mathbf{r}} | \mathbf{x}\text{-}\boldsymbol{\xi} | = - (1/4\pi) |\mathbf{x}\text{-}\boldsymbol{\xi}|^{-3} (r - r_1\cos\theta)$$

$$= - (1/4\pi) [\sqrt{r_1^2 + r^2 - 2rr_1\cos\theta}\ ]^{-3} (r - r_1\cos\theta)$$

$$= - (1/4\pi) \frac{r - r_1\cos\theta}{(r_1^2 + r^2 - 2rr_1\cos\theta)^{3/2}}$$

Now the problem is to take $x \to s$. That seems pretty easy to do

$$\lim_{\mathbf{x}\to\mathbf{s}} [\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(\mathbf{x}|\boldsymbol{\xi})] = - (1/4\pi) \frac{r - r\cos\theta}{(r^2 + r^2 - 2rr\cos\theta)^{3/2}}$$

$$= - (1/4\pi) \frac{r(1 - \cos\theta)}{(2r^2)^{3/2}(1-\cos\theta)^{3/2}} = - (1/4\pi) \frac{r}{2^{3/2}r^3(1-\cos\theta)^{1/2}}$$

$$= - (1/4\pi) \frac{1}{2^{3/2}r^2(1-\cos\theta)^{1/2}}$$

Now we have to remember that the great sphere is $r \to \infty$, so for $s \neq \xi$ we really do have

$$\lim_{\mathbf{x}\to\mathbf{s}} [\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(\mathbf{x}|\boldsymbol{\xi})] = 0$$

as required by our hopeful result that $\lim_{\mathbf{x}\to\mathbf{s}} [\partial_{\boldsymbol{\xi}\mathbf{n}}\, g(\mathbf{x}|\boldsymbol{\xi})] = - \delta^{(2)}(\mathbf{s}\text{-}\boldsymbol{\xi})$.

Now what does $\delta^{(2)}(\mathbf{s}\text{-}\boldsymbol{\xi})$ look like? Since $\mathbf{s} = r\hat{\mathbf{z}}$ we are in the region of the north pole, so we can make a little tangent 2D coordinate system there. In general we have

$$\xi_{\mathbf{x}} = r\sin\theta\cos\varphi \approx r\theta\cos\varphi$$
$$\xi_{\mathbf{y}} = r\sin\theta\sin\varphi \approx r\theta\sin\varphi$$
$$\xi_{\mathbf{z}} = r\cos\theta \approx r$$

Then I would say, since **s** is at the north pole exactly,

$$\delta^{(2)}(\mathbf{s}\text{-}\boldsymbol{\xi}) = \delta(\xi_{\mathbf{x}})(\xi_{\mathbf{y}}) = \delta(r\theta\cos\varphi)\delta(r\theta\sin\varphi) = \delta(x)\delta(y)$$

Now recall the notion that $\delta(x)\delta(y) = \delta(r)/(2\pi r)$ which you verify by integrating $\int dxdy = \int rdrd\theta$. Then I claim that

$$\delta(x)\delta(y) = \delta(r\theta\cos\varphi)\delta(r\theta\sin\varphi) = \delta(r\theta)/[2\pi r\theta]$$

since $(r\theta\cos\varphi)^2 + (r\theta\sin\varphi)^2 = (r\theta)^2$. Then my claim is

$$\delta^{(2)}(\mathbf{s}-\boldsymbol{\xi}) = \delta(r\theta)/[2\pi r\theta] = \delta(\theta)/[2\pi r^2\theta]$$

Again, if $\theta \neq 0$, since $r \to \infty$ on great sphere, we have $\delta^{(2)}(\mathbf{s}-\boldsymbol{\xi}) = 0$.

So what I need still to show is that, as $\theta \to 0$ (but also in the limit $r \to \infty$!!)

$$-(1/4\pi)\ \frac{1}{2^{3/2}r^2(1-\cos\theta)^{1/2}} = \delta(\theta)/[2\pi r^2\theta]$$

Now one thing I can do is say $\cos\theta = 1 - \theta^2/2$ so that $(1-\cos\theta) = \theta^2/2$ and then

$$-(1/4\pi)\ \frac{1}{2^{3/2}r^2(1-\cos\theta)^{1/2}} \approx -(1/4\pi)\ \frac{1}{2^{3/2}r^2(\theta/\sqrt{2})} = -(1/4\pi)\ \frac{1}{2r^2(\theta)} = -(1/2\pi)\ 1/(r^2\theta)$$

so now I want to show that

$$-(1/2\pi)\ 1/(r^2\theta) = \delta(\theta)/[2\pi r^2\theta]$$

or

$$-1 = \delta(\theta)$$

The problem here of course is that $1/4\pi R$ is NOT the correct Green g for a sphere of radius r. So to do this example right, you would have to use the correct g, and then I have more work. That is what I did in the 2D unit circle example above.

OK, I will let this rest some more. I think the real proof is in the original idea:

$$-\int_\sigma dS_\xi\ f(\xi)\ \partial_{\xi n} g(x|\xi) \to + f(x)$$

$$\Rightarrow \lim_{\xi \to x}\ \partial_{\xi n} g(x|\xi) = -\delta^{(2)}(\xi - x)$$

It still seems odd that Stak did not mention this anywhere. As you page through Chapters 5 and 6, the first time you see this kind of equation,

$$u_I(x) = \int_R d\xi\ g(x|\xi)\ q(\xi) - \int_\sigma dS_\xi\ f(\xi)\ \partial_{\xi n} g(x|\xi)\ .$$ // note second BC term

is on page 135 D and (6.81). Notice the definition of I(x|ξ) on page 136. So I would be looking for

$$\lim_{\xi \to x} I(x|\xi) = \delta^{(n-1)}(\xi - x)$$

Now in the special case that f(ξ) = 1 on an entire boundary with no sources we know that $u_I(x) = 1$ and I agree. Then we have

$$u_I(x) = \int_R d\xi\, g(x|\xi)\, q(\xi) - \int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) .$$

$$1 = 0 - \int_\sigma dS_\xi\, \partial_{\xi n} g(x|\xi)$$

or

$$\int_\sigma dS_\xi\, \partial_{\xi n} g(x|\xi) = 1 \qquad \text{for any x in R} \qquad \text{his (6.84)}$$

I agree with this result, but it is different from my claim which is this (whether or not q(x) = 0)

$$-\int_\sigma dS_\xi\, f(\xi)\, \partial_{\xi n} g(x|\xi) \to f(s) \qquad \text{as } x \to s \text{ with s on } \sigma$$

which in turn seems to imply that

$$-\partial_{\xi n} g(x|\xi) \to \delta^{(n-1)}(s - \xi)$$

as I page through the rest of Chapter 6, nothing like this ever appears. His other simpler book which I have djvu has cut all this stuff out so is not very helpful. I just don't know any other books that have this kind of information. But if you do amazon on "boundary value problems", there are other books:

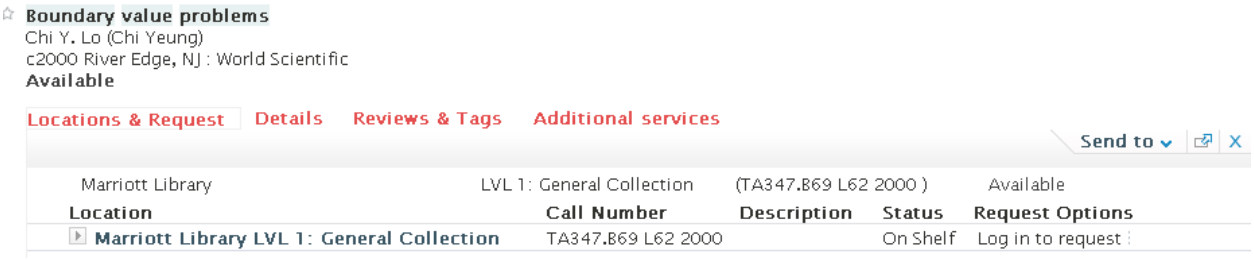
Note: all the stuff below is now copied into My Interest / Library Stuff/ a notes doc.



This search seems to reveal 2,729 real books with "boundary value problems" in the title!!! That is absolutely amazing really! Stak is just one (or two) of these books. A recent one is by David L. Powers. I was able to download this at scribd by uploading my blue Jackson errata document. The Powers book is 515 pages (5th Ed), but it just is not as advanced as Stak. Never does n dimensions for example. It has a lot of practical engineering problems in it.

I would probably take a lot of searching to find a book on the Stak level. Probably a library job.

If I scan Marriott for same phrase, there are 204 books, 146 not in ARC! Example



There are 64 books "available" right now. I cannot get them to list by call number! Three have "full text online" but they are not free to me of course.

Is there some other library which could make me a call number list? Marriott cannot do it, I remember asking. If I go to library of congress and do title keyword search on the phrase, I do get a call number list! They list 380 books. I set in some reasonable search limits. Then 183 books. Sort by author. But not all have call numbers! But not all books have my exact phrase in the title, so start over.

TITLES	
KTIL (Title: All) *Searches for words or an exact phrase in fields and subfields describing titles, series, subjects and authors.*	**Example:** **KTIL "canterbury tales"**
	Retrieves: catalog records for materials having the exact phrase *canterbury tales* in the title, including texts of and about "Canterbury Tales."

KSER (Series) *Searches for words or an exact phrase in titles of series.*	**Example:** **KSER canterbury** **Retrieves:** catalog records for materials in series, such as "Canterbury Papers Series" and "Canterbury Readers in Russian Poetry."

Well here is a KTIL expert search list. I will just write down call number areas that I see:

QA372
QA315 calc of variations
QA377 with Fourier Series
QA379 a lot of these
QA391 with elasticity
QA1 seems pure, Stak like books
QA3 with Markov
QA371
QA431
QA372 with DE's
QA374 with PDE
QC631 app iun E&M
QA401 in heat conduction but also general
QA931 with viscoelas
QC321 with heat
QA371 "in math phys" Stak is here

So the answer is that books are strewn around. The classification outline is here

http://www.loc.gov/catdir/cpso/lcco/

- A -- GENERAL WORKS - WP version - Word version
- B -- PHILOSOPHY. PSYCHOLOGY. RELIGION - WP version - Word version
- C -- AUXILIARY SCIENCES OF HISTORY - WP version - Word version
- D -- WORLD HISTORY AND HISTORY OF EUROPE, ASIA, AFRICA, AUSTRALIA, NEW ZEALAND, ETC. - WP version - Word version
- E -- HISTORY OF THE AMERICAS - WP version - Word version
- F -- HISTORY OF THE AMERICAS - WP version - Word version
- G -- GEOGRAPHY. ANTHROPOLOGY. RECREATION - WP version - Word version
- H -- SOCIAL SCIENCES - WP version - Word version
- J -- POLITICAL SCIENCE - WP version - Word version
- K -- LAW - WP version - Word version
- L -- EDUCATION - WP version - Word version
- M -- MUSIC AND BOOKS ON MUSIC - WP version - Word version
- N -- FINE ARTS - WP version - Word version
- P -- LANGUAGE AND LITERATURE - WP version - Word version
- Q -- SCIENCE - WP version - Word version
- R -- MEDICINE - WP version - Word version
- S -- AGRICULTURE - WP version - Word version
- T -- TECHNOLOGY - WP version - Word version
- U -- MILITARY SCIENCE - WP version - Word version
- V -- NAVAL SCIENCE - WP version - Word version
- Z -- BIBLIOGRAPHY. LIBRARY SCIENCE. INFORMATION RESOURCES (GENERAL) - WP version - Word version

I got the Q section as a PDF. The LOC refuses to reveal the classification details unless you pay!
The Classification for Q is a whole volume I see, Google has it for 1921. And archive.org has this same thing as a PDF again from 1921. It is a super slow PDF I guess due to aged paper, file itself is not very large. the DJVU converter takes 1 minute per page and there are 224 pages, so at least a 4 hour job. It is faster to look at this thing in google rather than the PDF!

This stuff simple is NOT on the web, I have to go to a library and take photos or copies!