A review of the generic regular BC problem PhL 8.10.11

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This subject is treated in Stak Chap 4. He does a long example, then addresses the general case on page 268. This is the 1D "Sturm-Liouville" problem in which various assumptions are made. My Chap 4 meta meta notes are not too bad and have all the major results, but they are not systematically derived there, something I will try to do here.

Our starting point is having some formally self-adjoint 2nd order differential operator L constructed from functions p and q, and then we can at least talk about this grid of equations:

operator:	$L_\lambda = L - \lambda s$	$L = -\partial_x(p\partial_x) + q$	
Fred general:	$L_\lambda u = f$	$Lu - \lambda su = f$	// inhomo 2
Set f = 0:	$L_\lambda \varphi_\lambda = 0$	$L\varphi_\lambda = \lambda s\varphi_\lambda$	// homo 2 = the EV problem
Set $\lambda = 0$:	$L_0 u = f$	$Lu = f$	// inhomo1
Set $f = \delta$:	$L_\lambda g = \delta$	$Lg - \lambda sg = \delta$	// Green's function problem with λ

It is required that functions $s(x)>0$ and $p(x)>0$ on our interval (a,b), q(x) is also real.

Lemma 0: An L of this form is formally self adjoint.

Proof:

$$\int_a^b dx\, v^*Lu = \int_a^b dx\, v^*[(pu')' + qu] = \int_a^b dx\, [v^*(pu')' + qv^*u] = \int_a^b dx\, (-v^{*\prime}(pu') + qv^*u)$$

$$= \int_a^b dx\, (-v^{*\prime}pu' + quv^*) = \int_a^b dx\, (+(v^{*\prime}p)'u + quv^*) = \int_a^b dx\, u[(v^{*\prime}p)' + qv^*]$$

$= \int_a^b dx\, u\,(Lv)^*$ $\quad\Rightarrow\ \langle v,Lu\rangle = \langle Lv,u\rangle$ if we ignore the parts

In the above we did two parts integrations and "threw out" the parts. The meaning of formally self adjoint is that $\langle v,Lu\rangle = \langle Lv,u\rangle$ if you ignore the parts.

Unmixed BC's and Full Self-adjointness of L (L is symmetric)

At each endpoint we have a standard unmixed homogeneous boundary condition meaning $Au(a) + Bu'(a) = 0$ where A and B must be real. The following Lemma will be useful at once, and is never obvious to me:

Lemma 1: If two functions u and v satisfy the BC at x=a, then $W[u,v;a] = W[u,\bar{v};a] = 0$.

Proof: $Au(a)+Bu'(a) = 0$, $Av(a)+Bv'(a) = 0$ $\Rightarrow$ $\begin{vmatrix} Au(a) & Bu'(a) \\ Av(a) & Bv'(a) \end{vmatrix} = 0$

$\Rightarrow AB \begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{vmatrix} = 0 \quad \Rightarrow W[u,v;a] = 0$

Since A and B are real, we can CC the second BC to get $A\bar{v}(a)+B\bar{v}'(a) = 0$, so $W[u,\bar{v};a] = 0$ as well. Since there is some similar BC at endpoint b, we know also that $W[u,v;b] = W[u,\bar{v};b] = 0$.

Corollary 1: If functions u satisfies an arbitrary unmixed BC at x=a, then $W[u,u;a] = W[u,\bar{u};a] = 0$. The first result is a 2=2 statement, but the second result is more interesting.

Lemma 2: Another less than obvious fact. For ANY functions u and v, we have

$$\bar{v}Lu - u\,\overline{Lv} = \partial_x \{\, p(x)\, W[u,\bar{v};x] \,\}$$

Proof: $L = -\partial_x(p\partial_x) + q \quad\Rightarrow\quad Lu = -(pu')'+qu$

$$v^*Lu - u(Lv)^* = v^*[-(pu')'+qu] - u[-(pv^*')'+qv^*] = -v^*(pu')' + u(pv^*')'$$

$$= -v^*[p'u'+pu''] + u[p'v^{*\prime}+pv^{*\prime\prime}] = p\,\{-v^*u'' + uv^{*\prime\prime}\} + p'\,\{-v^*u' + uv^{*\prime}\}$$

Now note that

$$W[u,v^*] = uv^{*\prime} - u'v^* \quad\Rightarrow$$
$$(W[u,v^*])' = u'v^{*\prime} + uv^{*\prime\prime} - u''v^* - u'v^* = uv^{*\prime\prime} - u''v^*$$

Thus we have shown above that

$$v^*Lu - u(Lv)^* = p\,\{-v^*u'' + uv^{*\prime\prime}\} + p'\,\{-v^*u' + uv^{*\prime}\} = p(W[u,v^*])' + p'\,W[u,v^*]$$
$$= \{\, p\, W[u,v^*]\,\}' \qquad \text{QED.}$$

If we now integrate Lemma2 from a to b, we get

$$\int_a^b dx \{ \bar{v}Lu - u\overline{Lv} \} = p(b)\, W[u,\bar{v}; b] - p(a)\, W[u,\bar{v}; a] = < v,Lu> - <Lv,u>$$

Now the pW stuff here is "the parts" and if we ignore the parts, we get <v,Lu> = <Lv,u>, so this is a second proof of Lemma 0 above which said that an L of our form was formally self adjoint. But now we apply Lemma 1 at the two end points and we find then that

$$\int_a^b dx \{ \bar{v}Lu - u\overline{Lv} \} = p(b)\, 0 - p(a)\, 0 = 0 = < v,Lu> - <Lv,u>$$

and now we see that <v,Lu> = <Lv,u> even if we keep the parts! So this is the first very major result! It says that in our BV problem situation, L is not only formally self adjoint, it is **fully self adjoint** (aka "**symmetric**") because "the parts" vanish.

Lemma 3: $[\, W(u,v)]' = W(u,v')$

Proof: We already derived this above in effect, but let's do it directly

$$[\, W(u,v)]' = [uv'-vu']' = u'v'+uv'' - v'u'-vu'' = uv'' -vu'' = W[u,v'] \quad \text{QED}$$

The Three Basic Theorems for Symmetric L's: real, real and orthogonal.

The following is shown on page 270 of Stak. I use the QM convention on inner products here.

Theorem 1: <Lu,u> is real. Proof: <u,Lu> = <Lu,u> = <u,Lu>*

Theorem 2: Eigenvalues of L are real

Proof: $L\varphi_\lambda = \lambda s\varphi_\lambda, \Rightarrow <\varphi_\lambda, L\varphi_\lambda> = \lambda <\varphi_\lambda, s\varphi_\lambda>$

$(L\varphi_\lambda)^* = \lambda^* s\varphi_\lambda^* \Rightarrow <\varphi_\lambda^*, (L\varphi_\lambda)^*> = \lambda^* <\varphi_\lambda^*, s\varphi_\lambda^*> = \lambda^* <s\varphi_\lambda, \varphi_\lambda> = \lambda^* <\varphi_\lambda, s\varphi_\lambda>$

where in 2nd last step we use <x*,y*>= <x,y>* = <y,x> and s=s*. Thus we have shown

$\lambda <\varphi_\lambda, s\varphi_\lambda> = <\varphi_\lambda, L\varphi_\lambda>$

$\lambda^* <\varphi_\lambda, s\varphi_\lambda> = <\varphi_\lambda^*, (L\varphi_\lambda)^*> = < L\varphi_\lambda, \varphi_\lambda>^* = < L\varphi_\lambda, \varphi_\lambda> = < \varphi_\lambda, L\varphi_\lambda>$

where we used Theorem 1 and then the fact that L is fully self-adjoint. Subtracting we find

$(\lambda - \lambda^*) <\varphi_\lambda, s\varphi_\lambda> = 0 \Rightarrow (\lambda - \lambda^*) = 0 \Rightarrow \lambda = \lambda^* \Rightarrow \lambda = \text{real}$

Theorem 3: Eigenfunctions of different λ are orthogonal with weight s

$\langle L\varphi_{\lambda'}, \varphi_\lambda \rangle = \langle \lambda' s\varphi_{\lambda'}, \varphi_\lambda \rangle = \lambda'^* \langle s\varphi_{\lambda'}, \varphi_\lambda \rangle = \lambda' \langle s\varphi_{\lambda'}, \varphi_\lambda \rangle$ by Theorem 2

$\langle \varphi_{\lambda'}, L\varphi_\lambda \rangle = \langle \varphi_{\lambda'}, \lambda s\varphi_\lambda \rangle = \lambda \langle s\varphi_{\lambda'}, \varphi_\lambda \rangle$

So we have shown

$\lambda' \langle s\varphi_{\lambda'}, \varphi_\lambda \rangle = \langle L\varphi_{\lambda'}, \varphi_\lambda \rangle = \langle \varphi_{\lambda'}, L\varphi_\lambda \rangle$ // since L is symmetric
$\lambda \langle s\varphi_{\lambda'}, \varphi_\lambda \rangle = \langle \varphi_{\lambda'}, L\varphi_\lambda \rangle$

Therefore

$(\lambda'-\lambda) \langle \varphi_{\lambda'}, L\varphi_\lambda \rangle = 0$

so of $\lambda \to \lambda'$, then $\langle \varphi_{\lambda'}, L\varphi_\lambda \rangle = 0$ QED.

Why g is symmetric

When we solve for g using the "usual method", we always write

$g(x|\xi) = A\, w(x_<)\, z(x_>)$ $x_< = \min(x,\xi)$ $x_> = \max(x,\xi)$

and we compute A from the jump condition, and we thereby construct g! This form works because we make sure function w satisfies the left BC and z the right BC. This is one reason why we wanted to have only unmixed BC's in our problem specification. Thus we can write

$g(x|\xi) = A\, w(\min(x,\xi)\, z(\max(x,\xi))$

Since the min and max functions are symmetric under $x \leftrightarrow \xi$, $g(x|\xi)$ must also be symmetric.

Solution u in terms of g.

Consider these two systems

$Lu-\lambda su = f$ $L_\lambda u = f$
$Lg-\lambda sg = \delta$ $L_\lambda g = \delta$

Let's conjecture that

$$u(x) = \int_a^b g(x|\xi;\lambda) f(\xi) d\xi$$

Try this in the first equation above LHS :

$$L_x \{ \int_a^b g(x|\xi;\lambda) f(\xi) d\xi \} - \lambda s(x) \{ \int_a^b g(x|\xi;\lambda) f(\xi) d\xi \}$$

$$= \{ \int_a^b L_x\, g(x|\xi;\lambda) f(\xi) d\xi \} - \lambda s(x) \{ \int_a^b g(x|\xi;\lambda) f(\xi) d\xi \}$$

$$= \{ \int_a^b [\lambda s(x)\, g(x|\xi;\lambda) + \delta(x-\xi\}]\, f(\xi) d\xi \} - \lambda s(x) \{ \int_a^b g(x|\xi;\lambda) f(\xi) d\xi \}$$

$$= \int_a^b \delta(x-\xi)\, f(\xi) d\xi = f(x)$$

Therefore our conjectured u(x) form solves our ODE AND since g meets the BC's, so does u, so u must be the solution. You might wonder about the possibility of having extra homo solutions floating around. They would be solutions to Lu-λsu = 0 or Lu = λsu. But if this is true, λ must be an eigenvalues. Thus, if we assume λ is NOT an EV, then there is no extra homo solution to add, and our conjectured form is the complete answer.

The Eigenfunction expansion for g

Consider

$$Lu-\lambda su = f = sF$$

where we have defined F ≡ f/s as shown. Then make these two expansions

$$u = \Sigma_n u_n \varphi_n \qquad F = \Sigma_n F_n \varphi_n$$

Then we have

$$(L-\lambda s)(\Sigma_n u_n \varphi_n) = s(\Sigma_n F_n \varphi_n) \quad \Rightarrow \quad \Sigma_n u_n (L-\lambda s)\varphi_n = s(\Sigma_n F_n \varphi_n)$$

$$\Rightarrow \Sigma_n u_n (\lambda_n s-\lambda s)\varphi_n = s\Sigma_n F_n \varphi_n \quad \Rightarrow \quad \Sigma_n u_n (\lambda_n-\lambda)\varphi_n = \Sigma_n F_n \varphi_n$$

we then appeal to the completeness of the φ_n to get

$$u_n(\lambda_n-\lambda) = F_n \quad \Rightarrow \quad u_n = F_n/(\lambda_n-\lambda)$$

As long as λ differs from all eigenvalues, we then have a solution to our f-driven problem

$$u(x) = \Sigma_n u_n \varphi_n(x) = \Sigma_n \varphi_n(x) F_n/(\lambda_n-\lambda)$$

Now consider the special case where f = δ. We then have

$$F(x) = \delta(x-\xi)/s(x)$$

$$F_n = \langle \varphi_n, sF \rangle = \langle \varphi_n, s\delta/s \rangle = \langle \varphi_n, \delta \rangle = \varphi_n(\xi)^*$$

We have then shown that

$$g(x) = g(x|\xi;\lambda) = \Sigma_n \varphi_n(x) F_n/(\lambda_n-\lambda) = \Sigma_n \varphi_n(\xi)^*\varphi_n(x) /(\lambda_n-\lambda)$$

or if we throw in the symmetry of g, we get

$$g(x|\xi;\lambda) = \Sigma_n \varphi_n(x)^*\varphi_n(\xi) /(\lambda_n-\lambda)$$

This is a very famous and important result, and I have shown exactly how you obtain it!

Location of the eigenvalues.

Stak claims on page 272 that there can be at most a finite number of negative λ_n but on the right they will march off to infinity and ∞ will be the only accumulation point. He proves this in Exercise 4.7 which includes a whole page of waypoints and which I never did, and I won't do it right here.

Completeness of the eigenfunctions

As Stak shows on page 272, you have to appeal to the theory of integral equations (after you convert the eigenvalue ODE $Lu=\lambda su$ to an integral equation) to conclude that the functions $\sqrt{s}\,\varphi_n$ form a complete orthonormal set, where one ingredient is that the kernel now $g(x|\xi)$ is symmetric. In our work above, we always include s in the inner product as $\langle f,sg\rangle$ and thus treat s as a weight function in the integral. If we say

$$f(x) = \Sigma_n a_n [\sqrt{s}\,\varphi_n] = \sqrt{s}\ (\Sigma_n a_n \varphi_n) \qquad \Rightarrow \qquad f/\sqrt{s} = \Sigma_n a_n \varphi_n$$

If we define a new function $F = f/\sqrt{s}$ then we have $F = \Sigma_n a_n \varphi_n$. We assumed f was an arbitrary function for our expansion, but then of course F can be considered an arbitrary function as well, since for any F we have $f = \sqrt{s}\ F$ which is then an arbitrary function. The point is that if $\sqrt{s}\,\varphi_n$ forms a complete set, then the functions φ_n also form a complete set, and we can use either complete set.

Now if we select the φ_n to be orthonormal, we then have

$$\langle\varphi_m, s\varphi_n\rangle = \delta_{n,m}$$

This tells us that the completeness relation must have this form

$$\delta(x-\xi)/s(x) = \Sigma_n \varphi_n^*(x)\varphi_n(\xi)$$

To verify this result, we take the weighed inner product against each eigenfunction of our complete basis

$$\langle\varphi_m,\ s(x)\delta(x-\xi)/s(x)\rangle = \langle\varphi_m,\ s(x),\ \Sigma_n \varphi_n^*(x)\varphi_n(\xi)\rangle$$

$\varphi_m(\xi) = \Sigma_n \varphi_n(\xi) \langle \varphi_m, s(x), \varphi_n^*(x) \rangle = \Sigma_n \varphi_n(\xi) \delta_{m,n} = \varphi_m(\xi)$

Since the projections of our two sides are the same for every element of a complete basis, the two sides must be equal.

The integral of g formula

We have already shown that

$g(x|\xi;\lambda) = \Sigma_n \varphi_n(x)^*\varphi_n(\xi) /(\lambda_n-\lambda)$

where the λ_n lie on the real axis. If we use a CCW great circle contour, it certainly will include that real axis, and we can then say

$(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_n \varphi_n(x)^*\varphi_n(\xi)\, (1/2\pi i) \int_C d\lambda/(\lambda_n-\lambda) = - \Sigma_n \varphi_n(x)^*\varphi_n(\xi)$

The contour is in the correct direction around each pole. The minus sign arise because the residue formula requires $1/(\lambda-\lambda_n)$ on the bottom!

We can now combine our integral of g formula with our completeness formula to say

$- (1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_n \varphi_n(x)^*\varphi_n(\xi) = \delta(x-\xi)/s(x) \qquad \langle \varphi_m, s\varphi_n \rangle = \delta_{n,m}$

We have now proven this to be true for any regular BV problem! A common method we use to *find* the eigenfunctions is to compute g using standard methods, and try to get it into a Σ_n form with a symmetric summand, and then we can just "read off" the $\varphi_n(x)$ normalized eigenfunctions!

Why the spectrum is discrete

Lemma 4: Abel's formula $W[u,v; x] = C e^{-m(x)}$.

We go way back to Abel's formula for the Wronskian on page 60, which applies to any second order L. I will prove this result right here. Consider

$L = a_0\partial_x^2 + a_1\partial_x + a_2 \qquad Lu = a_0u'' + a_1u' + a_2u$

Assume that Lu=0 and Lv = 0 so that u and v are both homo solutions. Then consider

$vLu-uLv = 0-0 = 0$

$v(a_0u'' + a_1u' + a_2u) - u(a_0v'' + a_1v' + a_2v) = 0$

$v(a_0u'' + a_1u') - u(a_0v'' + a_1v') = 0$

$a_0(v\,u'' - u\,v'') + a_1(v\,u' - u\,v') = 0$

$a_0\{W[v,u]\}' + a_1\,W[v,u] = 0$

where we have used Lemma 3 above. Since we could have done u↔v at the start, we must also have

$a_0\{W[u,v]\}' + a_1\,W[u,v] = 0$

This last first order ODE has the form

$a_0W'+a_1W = 0 \quad \Rightarrow \quad dW/dx = -(a_1/a_0)W \quad \Rightarrow \quad dW/W = -(a_1/a_0)dx$

$\Rightarrow \ln W = -\int^{x} dx'\,(a_1(x')/a_0(x')) + c \quad \Rightarrow \quad W = C\exp\left[-\int^{x} dx'\,(a_1(x')/a_0(x'))\right]$

So we have shown that

$W[u,v;\,x] = W = C\exp\left[-\int^{x} dx'\,(a_1(x')/a_0(x'))\right] = C\,e^{-m(x)}$

and this is Abel's baby, where C is some constant one could determine from solutions u and v.

Corollary 4: The function m(x) is finite for all x since we assume $a_0 \neq 0$. Thus, the only way for the Wronskian to vanish is if C = 0. Thus, if $W[u,v;\,x_0] \neq 0$, then W[u,v; x] for all x !