A review of the generic singular BC problem PhL 8.10.11

Begin scan of the "regular" doc and look for things that are now different. 1
Digression to review of my Stak Meta Notes on this subject 1
Resume Scan of the Regular doc, looking for differences 3
The Eigenfunction expansion for g 3
Completeness of the eigenfunctions 5
The integral of g formula 6

Begin scan of the "regular" doc and look for things that are now different.

I will now just naively read through the regular BC doc and look for things that might be a problem in the case that b= ∞.

I think formal self-adjointness is OK, since we ignore parts.
Lemma 1 is OK.
Lemma 2 is OK.

However, when we integrate Lemma 2 we get

$$\int_a^\infty dx \{ \bar{v}Lu - u\overline{Lv} \} = p(\infty) W[u,\bar{v};\infty] - p(a) W[u,\bar{v};a]$$

where recall u and v are ANY functions. We are concerned about those ∞ appearances in the above!

Digression to review of my Stak Meta Notes on this subject

Now we have to pay attention to the meta notes on this subject. The trick is to realize that there must exist two linearly independent functions φ and ψ of $L_\lambda u=0$ which have certain unmixed BC's at x=a. Each has a different BC there, but our only interest is in having two lin indep solutions of the ODE (without regard to the original problem BC's) with both $p(a)W(\varphi,\bar{\varphi};a) = 0$ and $p(a)W(\psi,\bar{\psi};a) = 0$. Stak shows that $p(a)W(\varphi,\psi;a) = -(\alpha_1^2 + \alpha_2^2) \neq 0$ and that is why they are lin indep. The BC's at a for φ and ψ are carefully selected to make this these two pW objects vanish and W(φ,ψ) not vanish.

Our concern is the integrability of a candidate solution at the b=∞ endpoint. We try u = φ + mψ as our candidate for a function which is finite-s-norm at the upper endpoint, and of course u and ψ are lin indep. We back off the limit $b_0 \to b = \infty$ and we select $\beta_{1,2}$ in order to get $p(b_o) W(u, \bar{u}; b_0) = 0$ from a right-end unmixed BC. Imposing this BC puts a condition on our parameter m. Since everything is a function of λ (things like φ,ψ and u, because they are all solutions of $L_\lambda f=0$), we find that m is also a function of λ, and we call it m(λ). We define $h = \beta_1/\beta_2$ from our imposed BC. As we let this ratio vary all it wants, the solution value for m(λ) lies on a circular locus in the m=plane. Here we are varying our b_0 BC in *every possible way* before taking the limit $b_0 \to b = \infty$. We want to see every possible way that we can obtain our desired result $p(b_o) W(u, \bar{u}; b_0) = 0$ at the right endpoint. Stak gives expressions for the center A and radius r of this circle. If λ = real the circle happens to be the real axis, so assume λ not real.

So, m lies on a certain circle in the complex m plane of some radius which is a function of b_0. As we take $b_0 \rightarrow b$, the limiting circle might have a finite radius (limit circle case) or a zero radius (limit point case). In the first case, it turns out that ψ is not finite-s-norm at the upper endpoint, but u is. In the circle case, both u and ψ are finite s norm, and we have two possible solutions at the upper endpoint. Now in either case, our one or two viable solutions satisfy p(b) W(u, $\bar{u}$; b) = 0.

Now we must keep in mind that everything hinges on our choice for parameter λ. For one λ we might in theory get limit circle, and for some other λ we might get limit point. But it turns out (Weyl's theorem) that if you are limit circle for one λ, then you are limit circle for all λ! So you are motivated to pick a simple λ to see which case you are in.

So how does this help us? Here is one payoff. We selected some arbitrary λ and said $L_\lambda f = 0$ for all three functions φ,ψ and $u = \varphi + m\psi$. Since we are ignoring our original problem BC's, λ can just be any complex number. But equation 4.80a is true for any solution of $L_\lambda f = 0$. We have just shown that for either our one limit-point solution u, or for our two limit circle solutions u and ψ, pW = 0 both at x=a and at x=b. Therefore 4.80a tells us that Im(λ) = 0. So we recover "eigenvalues must be real".

Question: Suppose there is only one limit-point solution $u = \varphi + m\psi$ for some m. Can we argue that the BC at x=a on u is *arbitrary*? Lets try to write

$$Au(a) + Bu'(a) = 0$$

$$A[\, \varphi(a) + m\psi(a)] + B[\, \varphi'(a) + m\psi'(a)] = 0$$

$$A[\, -\alpha_2 + m\alpha_1] + B[\, \alpha_1/p(a) + m\alpha_2/p(a)] = 0$$

$$\alpha_1[mA + B/p(a)] + \alpha_2[\, -A + Bm/p(a)] = 0$$

$$\alpha_1/\alpha_2 = [\, A - Bm/p(a)]/[\, mA + B/p(a)] = [\, (A/B) - m/p(a)]/[\, m(A/B) + 1/p(a)]$$

For a given m on the solution circle or point, the only problem we might have is if the ratio of A and B are such that m(A/B)= -1/p(a). If we assume that our BC avoids this particular ratio, then we can just pick any α_1 and α_2 such that α_1/α_2 = the RHS shown above. If we hit the bad ratio, we just select $\alpha_1 = 0$ and α_2 = any non-zero value. Remember the condition is that you cannot have both α_1 and $\alpha_2 = 0$ at the same time. So, I think we can claim that our homo unmixed BC at the left end is in fact arbitrary.

So we now look at our original singular problem. We have some arbitrary unmixed BC at the left end a. We know there are either 1 or 2 solutions of $L_\lambda f = 0$ that are integrable at the high end $b = \infty$. So consider the two cases:

Limit Point Case:

solution = u(x) meets arbitrary BC on the left, if finite s norm on the right

In this case, we know there IS a solution on our interval (a,∞) that meets our left end BC whatever (almost) it is. In effect, our right end BC is just that the function be finite s norm there, which surely means $u \rightarrow 0$ at the high end.

Limit Circle Case:

solutions = $u(x)$, $\psi(x)$

In this case, we can write a candidate general solution as $w(x) = c\,u(x)+d\,\psi(x)$. We are OK at the right end (we meet the effective BC there) for and c,d choice. Our left end BC is this

$$Aw(a) + Bw'(a) = 0$$

$$A[\, c\,u(a) + d\psi(a)] + B[\, c\,u'(a) + d\psi'(a)] = 0$$

$$c\,[A\,u(a) + B\,u'(a)\,] + d\,[\,A\psi(a) + B\psi'(a)] = 0$$

$$(c/d) = -\,[\,A\psi(a) + B\psi'(a)]/\,[A\,u(a) + B\,u'(a)\,] = -\,[\,(A/B)\psi(a) + \psi'(a)]/\,[(\,A/B)\,u(a) + u'(a)\,]$$

In general for arbitrary A and B, it is clear that we can find a ratio (c/d) that works! If the denominator happens to vanish then we just set c = 0 and d = anything. Notice that in all cases, the overall scale of any solution of $L_\lambda f=0$ is arbitrary.

Conclusion: In either case, circle or point, if we specify an unmixed BC at the low end a, there will be some unique solution to our singular BV problem for interval (a,∞) for given real λ unless λ happens to not be in the spectrum, in which case there will be no solution.

If we *ignore* our low end BC, we conclude that for general λ, there will be one or two viable solutions (finite s norm) based on limit or circle. But if λ = real, there might be no solutions!

Although Stak uses a low end BC in his Weyl proofs, he is mostly interested in deciding whether there are 1 or 2 finite-s-norm solutions at a given singular endpoint, without regard to a BC at the other endpoint. Indeed, you might have both endpoints be singular.

Resume Scan of the Regular doc, looking for differences

The Three Basic Theorems seem still valid, since I did not assume discrete spectrum there.
And g is still symmetric for the same reason.
Solution u in terms of g stays the same.

The Eigenfunction expansion for g

Here we do have to make changes. We know that the continuous part of any singular spectrum comes from coalescence of discrete points as $b\mathbf{0}\rightarrow b = \infty$. So in the limit we have to included potentially both a discrete and continuous contribution to an expansion, so we now write

$$u(x) = \Sigma_{\lambda=\lambda n}\, u_{\lambda n}\varphi_{\lambda n}(x) \;+ \int d\lambda'\, u_{\lambda'}\, \varphi_{\lambda'}(x)$$

$$F(x) = \Sigma_{\lambda=\lambda n}\, F_{\lambda n}\varphi_{\lambda n}(x) \;+ \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)$$

This is what a complete expansion now looks like in the general case. So here we go

$$Lu-\lambda su = f = sF$$

$$(L-\lambda s)\,[\Sigma_{\lambda=\lambda n}\, u_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, u_{\lambda'}\, \varphi_{\lambda'}(x)] = s\,[\Sigma_{\lambda=\lambda n}\, F_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)]$$

$$[\Sigma_{\lambda=\lambda n}\, u_{\lambda n}(L-\lambda s)\varphi_{\lambda n}(x) + \int d\lambda'\, u_{\lambda'}\, (L-\lambda s)\varphi_{\lambda'}(x)] = s\,[\Sigma_{\lambda=\lambda n}\, F_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)]$$

$$[\Sigma_{\lambda=\lambda n}\, u_{\lambda n}(\lambda_n s-\lambda s)\varphi_{\lambda n}(x) + \int d\lambda'\, u_{\lambda'}\, (\lambda' s-\lambda s)\varphi_{\lambda'}(x)] = s\,[\Sigma_{\lambda=\lambda n}\, F_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)]$$

$$[\Sigma_{\lambda=\lambda n}\, u_{\lambda n}(\lambda_n-\lambda)\varphi_{\lambda n}(x) + \int d\lambda'\, u_{\lambda'}\, (\lambda'-\lambda)\varphi_{\lambda'}(x)] = [\Sigma_{\lambda=\lambda n}\, F_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)]$$

$$\Sigma_{\lambda=\lambda n}\,[\, u_{\lambda n}(\lambda_n-\lambda) - F_{\lambda n}]\,\varphi_{\lambda n}(x) + \int d\lambda'\,[\, u_{\lambda'}\, (\lambda'-\lambda) - F_{\lambda'}\,]\varphi_{\lambda'}(x) = 0$$

Now in the usual way we appeal to completeness in each EF in our set and we conclude

$$u_{\lambda n}(\lambda_n-\lambda) - F_{\lambda n} = 0$$
$$u_{\lambda'}\, (\lambda'-\lambda) - F_{\lambda'} = 0$$

$$u_{\lambda n}(\lambda_n-\lambda) = F_{\lambda n}$$
$$u_{\lambda'}\, (\lambda'-\lambda) = F_{\lambda'}$$

$$u_{\lambda n} = F_{\lambda n}/\,(\lambda_n-\lambda)$$
$$u_{\lambda'} = F_{\lambda'}/(\lambda'-\lambda)$$

Remember that λ is our thing in L_λ. Our solution is therefore

$$u(x) = \Sigma_{\lambda=\lambda n}\, u_{\lambda n}\varphi_{\lambda n}(x) + \int d\lambda'\, u_{\lambda'}\, \varphi_{\lambda'}(x)$$

$$= \Sigma_{\lambda=\lambda n}\,[F_{\lambda n}/\,(\lambda_n-\lambda)]\varphi_{\lambda n}(x) + \int d\lambda'[F_{\lambda'}/(\lambda'-\lambda)]\,\varphi_{\lambda'}(x)$$

Now let's try our special case of $f = \delta$ as we did before.

$$F(x) = \delta(x-\xi)/s(x)$$

$$F_{\lambda n} = \langle\varphi_{\lambda n},\, sF\rangle = \langle\, \varphi_{\lambda n},s\delta/s\rangle = \langle\, \varphi_{\lambda n\ n},\delta\rangle = \varphi_{\lambda n}\,(\xi)^*$$
$$F_{\lambda} = \langle\varphi_{\lambda},\, sF\rangle = \langle\varphi_{\lambda},s\delta/s\rangle = \langle\varphi_{\lambda},\delta\rangle = \varphi_{\lambda}(\xi)^*$$

We have then shown that

$$g(x) = g(x|\xi;\lambda) = \Sigma_{\lambda=\lambda n}\,[F_{\lambda n}/(\lambda_n-\lambda)]\varphi_{\lambda n}(x) + \int d\lambda'[F_{\lambda'}/(\lambda'-\lambda)]\,\varphi_{\lambda'}(x)$$

$$= \Sigma_{\lambda=\lambda n}\,[\varphi_{\lambda n}(\xi)^*/(\lambda_n-\lambda)]\varphi_{\lambda n}(x) + \int d\lambda'[\varphi_{\lambda}(\xi)^*/(\lambda'-\lambda)]\,\varphi_{\lambda'}(x)$$

$$= \Sigma_{\lambda=\lambda n}\,\varphi_{\lambda n}(\xi)^*\,\varphi_{\lambda n}(x)/(\lambda_n-\lambda) + \int d\lambda'\varphi_{\lambda}(\xi)^*\,\varphi_{\lambda'}(x)/(\lambda'-\lambda)]$$

or if we throw in the symmetry of g, we get

$$g(x|\xi;\lambda) = \Sigma_{\lambda=\lambda n}\,\varphi_{\lambda n}(x)^*\,\varphi_{\lambda n}(\xi)/(\lambda_n-\lambda) + \int d\lambda'\varphi_{\lambda}(x)^*\,\varphi_{\lambda'}(\xi)/(\lambda'-\lambda)]$$

I did all this work just to show that there are no weird extra weight functions appearing in the integral.

Completeness of the eigenfunctions

Here we need some work as well. I think we can normalize the $\varphi_{\lambda n}(x)$ and the $\varphi_{\lambda}(x)$ [remember that these are in theory unrelated functions!] we will get

$$\langle\varphi_{\lambda m}, s\varphi_{\lambda n}\rangle = \delta_{n,m}$$
$$\langle\varphi_{\lambda}, s\varphi_{\lambda'}\rangle = \delta(\lambda-\lambda')$$
$$\langle\varphi_{\lambda}, s\varphi_{\lambda n}\rangle = 0$$

The last item is the most mysterious, but with the notion of the cut as limit of poles, it ought to work.

This tells us that the completeness relation must have this form

$$\delta(x-\xi)/s(x) = \Sigma_{\lambda n}\,\varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda'\varphi_{\lambda'}(x)^*\,\varphi_{\lambda'}(\xi)$$

Keep in mind that we have

$L_{\lambda}\,\varphi_{\lambda n} = s\,\lambda_n\,\varphi_{\lambda n}$ for λ_n in the point spectrum
$L_{\lambda}\,\varphi_{\lambda} = s\,\lambda\,\varphi_{\lambda}$ for λ in the continuous spectrum

so we never want to write the dummy integration variable as λ !

To verify this completeness result, we take the weighed inner product against each eigenfunction of our complete basis

$$\varphi_{\lambda m}(\xi) = \langle\varphi_{\lambda m}, s(x)\delta(x-\xi)/s(x)\rangle = \langle\varphi_{\lambda m}, s(x)\,[\Sigma_{\lambda n}\,\varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda'\varphi_{\lambda}(x)^*\,\varphi_{\lambda'}(\xi)]\rangle$$

$$= \langle\varphi_{\lambda m}, s(x)\,[\Sigma_{\lambda n}\,\varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi)] = \Sigma_{\lambda n}\,\varphi_{\lambda n}(\xi)\;\delta_{m,n} = \varphi_{\lambda m}(\xi)$$

and similarly for the other case we will verify that $\varphi_{\lambda}(\xi) = \varphi_{\lambda}(\xi)$. Notice that the third result above is essential.

The integral of g formula

We have already shown that (our expansion formula with a mixed spectrum)

$$g(x|\xi;\lambda) = \Sigma_{\lambda=\lambda n}\, \varphi_{\lambda n}(x)^*\, \varphi_{\lambda n}(\xi)/(\lambda_n-\lambda) + \int d\lambda' \varphi_{\lambda}(x)^*\, \varphi_{\lambda'}(\xi)/(\lambda'-\lambda)]$$

where the λ_n and λ' lie on the real axis. If we use a CCW great circle contour, it certainly will include that real axis, and we can then say

$$(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}(x)^* \varphi_{\lambda n}(\xi)\, (1/2\pi i) \int_C d\lambda/(\lambda_n-\lambda)$$

$$+ \int d\lambda' \varphi_{\lambda}(x)^*\, \varphi_{\lambda'}(\xi)\, (1/2\pi i) \int_C d\lambda/(\lambda'-\lambda)$$

$$= -\Sigma_{\lambda n}\, \varphi_{\lambda n}(x)^* \varphi_{\lambda n}(\xi) - \int d\lambda' \varphi_{\lambda}(x)^*\, \varphi_{\lambda'}(\xi)$$

The contour is in the correct direction around each pole. The minus sign arise because the residue formula requires $1/(\lambda-\lambda_n)$ and $(\lambda-\lambda')$ on the bottom!

We can now combine our integral of g formula with our completeness formula to say

$$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}^*(x) \varphi_{\lambda n}(\xi) + \int d\lambda' \varphi_{\lambda}(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

We have now proven this to be true for any *singular* BV problem! A common method we use to *find* the eigenfunctions is to compute g using standard methods, and try to get it into a $\Sigma_n + \int$ form with a symmetric summand, and then we can just "read off" the $\varphi_n(x)$ and $\varphi_{\lambda}(x)$normalized eigenfunctions! I have never seen this done in the mixed case, but it ought to work.

I like writing the integration variable as λ' because it reminds me that it really is the eigenvalue parameter itself, not something related to it. That is to say, $\varphi_{\lambda'}(x)$ satisfies $L_{\lambda'}\, \varphi_{\lambda'}(x) = 0$. Sometimes people replace λ' by ν, but then this leads to confusion if we happen to have ν with meaning $\lambda = \nu^2$ say. The "measure" in the integral part above is just $d\lambda'$, not something else.

Summary of the Above

Assume a mixed spectrum, and assume **orthonormal eigenfunctions**, which means this

$$\langle \varphi_{\lambda m}, s\varphi_{\lambda n} \rangle = \delta_{n,m}$$
$$\langle \varphi_{\lambda}, s\varphi_{\lambda'} \rangle = \delta(\lambda-\lambda')$$

$<\varphi_{\lambda}, s\varphi_{\lambda n}> = 0$

Notice that the inner product associated with our Hilbert Space contains the weight function s !

These eigenfunctions are eigenfunctions of self-adjoint operator

$L_{\lambda} = L - \lambda s(x)$ // as on p 268

so

$L_{\lambda}\, \varphi_{\lambda n} = s\, \lambda_n\, \varphi_{\lambda n}$ for λ_n in the point spectrum

$L_{\lambda}\, \varphi_{\lambda} = s\, \lambda\, \varphi_{\lambda}$ for λ in the continuous spectrum

Theorem 1: Assuming the usual stuff, this set of eigenfunctions is complete, and the proper way to express this fact is (a one line proof is given above)

$$\delta(x-\xi)/s(x) = \Sigma_{\lambda n}\, \varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi)$$

Theorem 2: Bilinear expansion of g

Consider this Fred 2 inhomo

$$L_{\lambda}u = Lu-\lambda su = f = sF$$

We show above that the solution to this problem is

$$u(x) = \Sigma_{\lambda=\lambda n}\, [F_{\lambda n}/ (\lambda_n-\lambda)]\varphi_{\lambda n}(x) + \int d\lambda' [F_{\lambda'}/(\lambda'-\lambda)]\, \varphi_{\lambda'}(x)$$

$$\text{where } F(x) = \Sigma_{\lambda=\lambda n}\, F_{\lambda n}\, \varphi_{\lambda n}(x) + \int d\lambda'\, F_{\lambda'}\, \varphi_{\lambda'}(x)$$

We then *apply this* to the case f = δ and find F = δ/s so that $F_{\lambda n} = \varphi_{\lambda n}(\xi)^*$ etc and we find:

$$L_{\lambda}g = Lg-\lambda sg = \delta$$

$$g = g(x|\xi) = \Sigma_{\lambda=\lambda n}\, \varphi_{\lambda n}(\xi)^*\, \varphi_{\lambda n}(x)/ (\lambda_n-\lambda) + \int d\lambda' \varphi_{\lambda}(\xi)^*\, \varphi_{\lambda'}(x)/(\lambda'-\lambda)]$$

This is the bilinear expansion for g.

Theorem 3: Integral of g relation.

$$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi)$$

Combined Results

We can combine all these results into the following dense form:

$$-(1/2\pi i)\int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\,\varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda'\varphi_{\lambda'}(x)^*\,\varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

$$\langle\varphi_{\lambda m}, s\varphi_{\lambda n}\rangle = \delta_{n,m} \qquad \langle\varphi_{\lambda}, s\varphi_{\lambda'}\rangle = \delta(\lambda-\lambda') \qquad \langle\varphi_{\lambda}, s\varphi_{\lambda n}\rangle = 0$$

Notice carefully that in the integral, the integration variable is the actual eigenvalue variable.

We can state this symbolically where we include the integral with the sum

$$-(1/2\pi i)\int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\,\varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) = \delta(x-\xi)/s(x)$$

$$\langle\varphi_{\lambda m}, s\varphi_{\lambda n}\rangle = \delta_{n,m}$$

Notice there is no mysterious factor K_n as I have in some of my earlier quotes of this result.

Example 1: the KL transform business

We start with

$$-(xg')' + \mu xg - \lambda x^{-1} g = \delta(x-\xi)$$

$$L_\lambda = L - \lambda s \qquad s = x^{-1} \qquad L = -(xg')' + \mu xg$$

We show (see KL doc) that the free-space solution g is not unique, but that this is a good one to use:

$$g(x|\xi;\lambda) = I_{-i\gamma}(kx_<)\, K_{-i\gamma}(kx_>) \qquad \text{where } \gamma \equiv \sqrt{\lambda}\ ,\ \mu = k^2$$

Since this is a purely continuous spectrum problem, we know that

$$-(1/2\pi i)\int_C d\lambda\, g(x|\xi;\lambda) = \int d\lambda'\varphi_{\lambda}(x)^*\,\varphi_{\lambda'}(\xi)$$

We show in that same KL doc that

$$-\oint d\lambda\, g(x|\xi;\lambda) = \int_0^\infty d\lambda'\,(2/\pi)\sin(\pi\alpha)K_\alpha(kx)K_\alpha(k\xi) \qquad \text{where } \alpha = i\sqrt{\lambda'}$$

To be λ purist, we rewrite

$$K_\alpha(x) = K_{i\sqrt{\lambda'}}(x) \qquad \sin(\pi\alpha) = \sin(\pi i\sqrt{\lambda'}) = i\,\mathrm{sh}(\pi\sqrt{\lambda'})$$

so the above becomes

$$- \oint d\lambda\, g(x|\xi;\lambda) \;=\; \int_0^\infty d\lambda' \,(2i/\pi)\, \mathrm{sh}(\pi\sqrt{\lambda'}\,)\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi)$$

$$- (1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) \;= (1/\pi^2) \int_0^\infty d\lambda' \,\mathrm{sh}(\pi\sqrt{\lambda'}\,)\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi)$$

Therefore we may directly identify

$$\int_0^\infty d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi) = \;(1/\pi^2) \int_0^\infty d\lambda' \,\mathrm{sh}(\pi\sqrt{\lambda'}\,)\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi)$$

The integrands must be equal, so we get

$$\varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi) \;= (1/\pi^2)\, \mathrm{sh}(\pi\sqrt{\lambda'}\,)\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi)$$

We then have this completely unambiguous expression for the eigenfunction (remember the order symmetry of the K, and remember that $K_{i\sqrt{\lambda'}}(kx)$ is real.

$$\varphi_\lambda(x) = \;(1/\pi)\, \sqrt{\mathrm{sh}(\pi\sqrt{\lambda}\,)}\;\, K_{i\sqrt{\lambda}}(kx)$$

Now at this point we can use $\gamma \equiv \sqrt{\lambda}$ or $\lambda = \gamma^2$, and choose to label φ with γ, so we have

$$\varphi_\gamma(x) = \;(1/\pi)\, \sqrt{\mathrm{sh}(\pi\gamma)}\;\, K_{i\gamma}(kx)$$

Then we have from above $\qquad\qquad \lambda = \gamma^2 \quad d\lambda = 2\gamma d\gamma$

$$- (1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) \;= (1/\pi^2) \int_0^\infty d\lambda' \,\mathrm{sh}(\pi\sqrt{\lambda'}\,)\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi)$$

$$= (1/\pi^2) \int_0^\infty [\,2\gamma d\gamma]\, \mathrm{sh}(\pi\gamma)\, K_{i\gamma}(kx)\, K_{i\gamma}(k\xi)$$

$$= (2/\pi^2) \int_0^\infty d\gamma\, \gamma\, \mathrm{sh}(\pi\gamma)\, K_{i\gamma}(kx)\, K_{i\gamma}(k\xi)$$

$$= \delta(x-\xi)/s(x) \;= x\, \delta(x-\xi) \qquad\qquad \text{// agrees with 4.137 Stak Vol I}$$

Now we can look at orthogonality:

$$\langle \varphi_\lambda, s\varphi_{\lambda'} \rangle \;= \delta(\lambda-\lambda')$$

$$\int_0^\infty dx\; s(x)\, (1/\pi)\, \sqrt{\mathrm{sh}(\pi\sqrt{\lambda}\,)}\;\, K_{i\sqrt{\lambda}}(kx)\, (1/\pi)\, \sqrt{\mathrm{sh}(\pi\sqrt{\lambda'}\,)}\;\, K_{i\sqrt{\lambda'}}(kx) \;= \delta(\lambda-\lambda')$$

or

$$(1/\pi^2)\sqrt{\sinh(\pi\sqrt{\lambda})}\sqrt{\sinh(\pi\sqrt{\lambda'})}\int_0^\infty dx\, s(x)\, K_{i\sqrt{\lambda}}(kx)\, K_{i\sqrt{\lambda'}}(kx) = \delta(\lambda-\lambda')$$

or

$$\int_0^\infty dx\, x^{-1}\, K_{i\sqrt{\lambda}}(x)\, K_{i\sqrt{\lambda'}}(x) = \pi^2\, \delta(\lambda-\lambda')' / \sinh(\pi\sqrt{\lambda})$$

If we now again write $\lambda = \gamma^2$ we have

$$\delta(\lambda-\lambda') = \delta(\gamma^2 - \gamma'^2) = \delta(\gamma-\gamma')/|2\gamma| = \delta(\gamma-\gamma')/(2\gamma)$$

Then orthogonality can be written, since $\gamma = \sqrt{\lambda}$,

$$\int_0^\infty dx\, x^{-1}\, K_{i\gamma}(kx)\, K_{i\gamma'}(kx) = \pi^2 \{ \quad \delta(\gamma-\gamma')/(2\gamma) \} / \sinh(\pi\gamma)$$

$$\int_0^\infty dx\, x^{-1}\, K_{i\gamma}(kx)\, K_{i\gamma'}(kx) = \delta(\gamma-\gamma')\, \pi^2/[2\gamma \sinh(\pi\gamma)]$$

Where can I verify this? You can see why the RHS is independent of k, just change variables on the LHS to y = kx and it then looks the same with k =1. I don't think Stak writes this down anywhere.

Amazingly I find this in a 2010 short journal article which I have saved as bielski.pdf

1. Introduction

In recent papers, Yakubovich [1] and Passian et al. [2] proved the following orthogonality relation for the Macdonald functions of imaginary order:

$$\int_0^\infty dx\, \frac{K_{i\nu}(x) K_{i\nu'}(x)}{x} = \frac{\pi^2}{2\nu \sinh(\pi\nu)} \delta(\nu-\nu') \quad (\nu, \nu' > 0). \tag{1.1}$$

and there is my formula!!!! The two referenced papers are 2006 and 2009. In this paper the authors more or less do what I just did above and in fact quote Stakgold 1998. So there is my verification!

Now here is the transform taken from my KL doc

$f(x) = (2/\pi^2) \int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, F(\nu)$ which is 4.139.

$F(\gamma) \equiv \int_0^\infty dx\, f(x)\, K_{i\gamma}(kx)\, /x$

Finally I am going to attempt to gather the basic four results in one place and then try to convert it all to generic variable names. In this Example we have shown that

$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi)$ where $\gamma \equiv \sqrt{\lambda}$, $\mu = k^2$

$L_\lambda = L - \lambda s \qquad s = x^{-1} \qquad L = -(xg')' + \mu x g$

$\varphi_\lambda(x) = (1/\pi)\sqrt{\sinh(\pi\sqrt{\lambda})}\ K_{i\sqrt{\lambda}}(kx)$

$\varphi_\gamma(x) = (1/\pi)\sqrt{\sinh(\pi\gamma)}\ K_{i\gamma}(kx) \qquad \gamma = \sqrt{\lambda}$

$x\,\delta(x-\xi) = (1/\pi^2)\int_0^\infty d\lambda' \sinh(\pi\sqrt{\lambda'})\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, K_{i\gamma}(k\xi)$

$\int_0^\infty dx\, x^{-1} K_{i\gamma}(kx)\, K_{i\gamma'}(kx) = \delta(\gamma-\gamma')\,\pi^2/[2\gamma \sinh(\pi\gamma)]$

$f(x) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, F(\nu)$ which is 4.139.

$F(\gamma) \equiv \int_0^\infty dx\, f(x)\, K_{i\gamma}(kx)/x$

So one way to do all this stuff is to write $\mu = k^2$ in the ODE and this remove μ completely:

$-(xg')' + k^2 x g - \lambda x^{-1} g = \delta(x-\xi)$ where $\gamma \equiv \sqrt{\lambda}$, $\mu = k^2$

$L_\lambda = L - \lambda s \qquad s = x^{-1} \qquad L = -(xg')' + k^2 x g$

$\varphi_\lambda(x) = (1/\pi)\sqrt{\sinh(\pi\sqrt{\lambda})}\ K_{i\sqrt{\lambda}}(kx)$

$\varphi_\gamma(x) = (1/\pi)\sqrt{\sinh(\pi\gamma)}\ K_{i\gamma}(kx) \qquad \gamma = \sqrt{\lambda}$

$x\,\delta(x-\xi) = (1/\pi^2)\int_0^\infty d\lambda' \sinh(\pi\sqrt{\lambda'})\, K_{i\sqrt{\lambda'}}(kx)\, K_{i\sqrt{\lambda'}}(k\xi) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, K_{i\gamma}(k\xi)$

$\int_0^\infty dx\, x^{-1} K_{i\gamma}(kx)\, K_{i\gamma'}(kx) = \delta(\gamma-\gamma')\,\pi^2/[2\gamma \sinh(\pi\gamma)]$

$f(x) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, F(\gamma)$ which is 4.139.

$F(\gamma) \equiv \int_0^\infty dx\, f(x)\, K_{i\gamma}(kx)/x$

Now why not just set $\lambda = \gamma^2$ in the ODE and then be rid of λ (now that we are done using it)

$-(xg')' + k^2 x g - \gamma^2 x^{-1} g = \delta(x-\xi)$

$L_\lambda = L - \gamma^2 s \qquad s = x^{-1} \qquad L = -(xg')' + k^2 x g$

$\varphi_\gamma(x) = (1/\pi)\sqrt{\sinh(\pi\gamma)}\ K_{i\gamma}(kx)$

$x\,\delta(x-\xi) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, K_{i\gamma}(k\xi)$

$\int_0^\infty dx\, x^{-1} K_{i\gamma}(kx)\, K_{i\gamma'}(kx) = \delta(\gamma-\gamma')\,\pi^2/[2\gamma \sinh(\pi\gamma)]$

$f(x) = (2/\pi^2)\int_0^\infty d\gamma\, \gamma \sinh(\pi\gamma)\, K_{i\gamma}(kx)\, F(\gamma)$

$F(\gamma) \equiv \int_0^\infty dx\, f(x)\, K_{i\gamma}(kx)/x$

Now make replacements $k^2 = \alpha^2$ and $\gamma^2 = \beta^2$

$-(xu')' + \alpha^2 xu - \beta^2 x^{-1} u = 0$ $\quad L_\lambda u = Lu - \beta^2 s\, u = -(xu')' + \alpha^2 xu - \beta^2 x^{-1} u = 0$

$L_\lambda = L - \beta^2 s \quad s = x^{-1} \quad L = -(xu')' + \alpha^2 xu$

$\varphi_\beta(x) = (1/\pi)\sqrt{\mathrm{sh}(\pi\beta)}\; K_{i\beta}(\alpha x)$ $\quad L\,\varphi_\beta(x) = \beta^2 (x^{-1})\,\varphi_\beta(x)$

$x\,\delta(x-\xi) = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, K_{i\beta}(\alpha\xi)$

$\int_0^\infty dx\, x^{-1} K_{i\beta}(\alpha x)\, K_{i\beta'}(\alpha x) = \delta(\beta-\beta')\, \pi^2/[2\beta\, \mathrm{sh}(\pi\beta)]$

$f(x) = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \sinh(\pi\beta)\, K_{i\beta}(\alpha x)\, F(\beta)$

$F(\beta) \equiv \int_0^\infty dx\, f(x)\, K_{i\beta}(\alpha x)/x$

Then maybe reorder this into my usual ordering

$-(xu')' + \alpha^2 xu - \beta^2 x^{-1} u = 0$ or $Lu = \beta^2 x^{-1} u$ where $L = -(xu')' + \alpha^2 xu$

or $L_\lambda u = 0$ where $L_\lambda = L - \beta^2 x^{-1}$ $\quad \lambda = \beta^2 \quad s(x) = x^{-1}$

$\varphi_\beta(x) = (1/\pi)\sqrt{\mathrm{sh}(\pi\beta)}\; K_{i\beta}(\alpha x)$

$f(x) = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, F_\alpha(\beta)$ // expansion

$F_\alpha(\beta) \equiv \int_0^\infty dx\, x^{-1} f(x)\, K_{i\beta}(\alpha x)$ // projection

$\int_0^\infty dx\, x^{-1} K_{i\beta}(\alpha x)\, K_{i\beta'}(\alpha x) = \delta(\beta-\beta')\, \pi^2/[2\beta\mathrm{sh}(\pi\beta)]$ // orthogonality

$x\,\delta(x-x') = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, K_{i\beta}(\alpha x')$ // completeness

Example 2: the Hankel transform business

We start with (see Hankel doc)

$-(xw')' + \nu^2\, w/x - \lambda x w = 0 \quad \lambda = k^2 \quad k = \sqrt{\lambda}$

$Lg - \lambda s(x) g = \delta(x-\xi) \quad Lg = -(xg')' + \nu^2\, g/x \quad s(x) = x \quad x \text{ in } (0,\infty)$

We write the Green's Function problem and then solve it to get $g(x|\xi;\lambda) = (i\pi/2)\, J_\nu(\sqrt{\lambda}\, x_<)\, H_\nu^{(1)}(\sqrt{\lambda}\, x_>)$. I then show that

$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = (1/2) \int_0^\infty d\lambda'\; J_\nu(\sqrt{\lambda'}\, x_<) J_\nu(\sqrt{\lambda'}\, x_>)$

Direct comparison to

$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \int d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$

tells us two things. First,

$$(1/2)\ \int_0^\infty d\lambda'\ J_\nu(\sqrt{\lambda'}\, x_<)J_\nu(\sqrt{\lambda'}\, x_>) = \delta(x-\xi)/x$$

Second

$$\int d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi) = (1/2)\ \int_0^\infty d\lambda'\ J_\nu(\sqrt{\lambda'}\, x_<)J_\nu(\sqrt{\lambda'}\, x_>)$$

so we conclude from this that

$$\varphi_\lambda(x) = (1/\sqrt{2})\, J_\nu(\sqrt{\lambda}\, x)$$

We can now relable this orthonormal eigenfunction with $k = \sqrt{\lambda}$ $k^2 = \lambda$ $2kdk = d\lambda$

$$\varphi_k(x) = (1/\sqrt{2})\, J_\nu(kx)$$

Then completeness above becomes

$$(1/2)\ \int_0^\infty [2k'dk']\ J_\nu(k'x)J_\nu(k'\xi) = \delta(x-\xi)/x$$

or

$$\int_0^\infty dk'\, k'\, J_\nu(k'x)J_\nu(k'\xi) = \delta(x-\xi)/x$$

or

$$\int_0^\infty dk\, k\, J_\nu(kx)J_\nu('\xi) = \delta(x-\xi)/x$$

This equation I know is part of this data set

$$f(x) = \int_0^\infty dk\, k\, J_\nu(kx)\, F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty dx\, x\, J_\nu(kx)\, f(x) \qquad \text{// projection}$$

$$\int_0^\infty dx\, x\, J_\nu(kx)\, J_\nu(k'x) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\, k\, J_\nu(kx)\, J_\nu(kx') = \delta(x-x')/x \qquad \text{// completeness}$$

So now I will build the larger data set similar to the above

$$-(xu')' + \nu^2\, u/x - k^2 xu = 0 \quad \text{or} \quad Lu = \lambda xu \quad \text{where } L = -(xu')' + \nu^2\, u/x$$
$$\text{or} \quad L_\lambda u = 0 \quad \text{where } L_\lambda = L - \lambda x \quad \lambda = k^2 \quad s(x) = x$$

$\varphi_k(x) = (1/\sqrt{2})\, J_\nu(kx)$

$f(x) = \int_0^\infty dk\, k\, J_\nu(kx)\, F_\nu(k)$ // expansion

$F_\nu(k) = \int_0^\infty dx\, x\, J_\nu(kx)\, f(x)$ // projection

$\int_0^\infty dx\, x\, J_\nu(kx)\, J_\nu(k'x) = \delta(k-k')/k$ // orthogonality

$\int_0^\infty dk\, k\, J_\nu(kx)\, J_\nu(kx') = \delta(x-x')/x$ // completeness

Now replace $\nu^2 = b^2\, k^2 = a^2$ and rewrite it all

$-(xu')' + b^2\, x^{-1}u - a^2xu = 0$ or $Lu = a^2xu$ where $L = -(xu')' + b^2\, u/x$
or $L_\lambda u = 0$ where $L_\lambda = L - a^2x$ $\lambda = a^2$ $s(x) = x$

$\varphi_a(x) = (1/\sqrt{2})\, J_b(ax)$ // normalized eigenfunction

$f(x) = \int_0^\infty da\, a\, J_b(ax)\, F_b(a)$ // expansion

$F_b(a) = \int_0^\infty dx\, x\, J_b(ax)\, f(x)$ // projection

$\int_0^\infty dx\, x\, J_b(ax)\, J_b(a'x) = \delta(a-a')/k$ // orthogonality

$\int_0^\infty dk\, k\, J_b(ax)\, J_b(ax') = \delta(x-x')/x$ // completeness

and right here I will replicate the above KL block for comparison

$-(xu')' + \alpha^2xu - \beta^2x^{-1}u = 0$ or $Lu = \beta^2x^{-1}u$ where $L = -(xu')' + \alpha^2xu$
or $L_\lambda u = 0$ where $L_\lambda = L - \beta^2x^{-1}$ $\lambda = \beta^2$ $s(x) = x^{-1}$

$\varphi_\beta(x) = (1/\pi)\sqrt{\mathrm{sh}(\pi\beta)}\; K_{i\beta}(\alpha x)$ // normalized eigenfunction

$f(x) = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, F_\alpha(\beta)$ // expansion

$F_\alpha(\beta) \equiv \int_0^\infty dx\, x^{-1}\, f(x)\, K_{i\beta}(\alpha x)$ // projection

$\int_0^\infty dx\, x^{-1}\, K_{i\beta}(\alpha x)\, K_{i\beta'}(\alpha x) = \delta(\beta-\beta')\, \pi^2/[2\beta\mathrm{sh}(\pi\beta)]$ // orthogonality

$x\, \delta(x-x') = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, K_{i\beta}(\alpha x')$ // completeness